

# Elliptic Curves

(PARI-GP version 2.13.0)

An elliptic curve is initially given by 5-tuple  $v = [a_1, a_2, a_3, a_4, a_6]$  attached to Weierstrass model or simply  $[a_4, a_6]$ . It must be converted to an *ell* struct.

Initialize *ell* struct over domain  $D$      **E = ellinit**( $v, \{D = 1\}$ )  
over **Q**      $D = 1$   
over **F<sub>p</sub>**      $D = p$   
over **F<sub>q</sub>**,  $q = p^f$       $D = \text{ffgen}([p, f])$   
over **Q<sub>p</sub>**, precision  $n$       $D = O(p^n)$   
over **C**, current bitprecision      $D = 1.0$   
over number field  $K$       $D = nf$

Points are  $[x, y]$ , the origin is  $[0]$ . Struct members accessed as **E.member**:

- All domains: **E.a1, a2, a3, a4, a6, b2, b4, b6, b8, c4, c6, disc, j**
- $E$  defined over **R** or **C**  
 $x$ -coords. of points of order 2     **E.roots**  
periods / quasi-periods     **E.omega, E.eta**  
volume of complex lattice     **E.area**

- $E$  defined over **Q<sub>p</sub>**  
residual characteristic     **E.p**  
If  $|j|_p > 1$ : Tate's  $[u^2, u, q, [a, b], \mathcal{L}]$      **E.tate**
- $E$  defined over **F<sub>q</sub>**  
characteristic     **E.p**  
 $\#E(\mathbf{F}_q)/\text{cyclic structure/generators}$      **E.no, E.cyc, E.gen**

- $E$  defined over **Q**  
generators of  $E(\mathbf{Q})$  (require **elldata**)     **E.gen**  
 $[a_1, a_2, a_3, a_4, a_6]$  from  $j$ -invariant     **ellfromj(j)**  
cubic/quartic/biquadratic to Weierstrass     **ellfromeqn(eq)**  
add points  $P + Q / P - Q$      **elladd(E, P, Q), ellsub**

- negate point     **ellneg(E, P)**
- compute  $n \cdot P$      **ellmul(E, P, n)**
- check if  $P$  is on  $E$      **ellisoncurve(E, P)**
- order of torsion point  $P$      **ellorder(E, P)**
- $y$ -coordinates of point(s) for  $x$      **ellordinate(E, x)**
- $[\phi(z), \phi'(z)] \in E(\mathbf{C})$  attached to  $z \in \mathbf{C}$      **ellztopoint(E, z)**
- $z \in \mathbf{C}$  such that  $P = [\phi(z), \phi'(z)]$      **ellpointtoz(E, P)**
- $z \in \bar{\mathbf{Q}}^*/q\mathbf{Z}$  to  $P \in E(\bar{\mathbf{Q}}_p)$      **ellztopoint(E, z)**
- $P \in E(\bar{\mathbf{Q}}_p)$  to  $z \in \bar{\mathbf{Q}}^*/q\mathbf{Z}$      **ellpointtoz(E, P)**

- Change of Weierstrass models, using**  $v = [u, r, s, t]$   
change curve  $E$  using  $v$      **ellchangecurve(E, v)**  
change point  $P$  using  $v$      **ellchangept(P, v)**  
change point  $P$  using inverse of  $v$      **ellchangeptinv(P, v)**

- Twists and isogenies**  
quadratic twist     **elltwist(E, d)**  
 $n$ -division polynomial  $f_n(x)$      **elldivpol(E, n, \{x\})**  
 $[n]P = (\phi_n \psi_n : \omega_n : \psi_n^2)$ ; return  $(\phi_n, \psi_n^2)$      **ellxn(E, n, \{x\})**  
isogeny from  $E$  to  $E/G$      **ellisogeny(E, G)**  
apply isogeny to  $g$  (point or isogeny)     **ellisogenyapply(f, g)**  
torsion subgroup with generators     **elltors(E)**

- Formal group**  
formal exponential,  $n$  terms     **ellformalexp(E, \{n\}, \{x\})**  
formal logarithm,  $n$  terms     **ellformallog(E, \{n\}, \{x\})**  
 $\log_E(-x(P)/y(P)) \in \mathbf{Q}_p$ ;  $P \in E(\mathbf{Q}_p)$      **ellpadiclog(E, p, n, P)**  
 $P$  in the formal group     **ellformalpoint(E, \{n\}, \{x\})**  
 $[\omega/dt, x\omega/dt]$      **ellformaldifferential(E, \{n\}, \{x\})**  
 $w = -1/y$  in parameter  $-x/y$      **ellformalw(E, \{n\}, \{x\})**

## Curves over finite fields, Pairings

random point on  $E$      **random(E)**  
 $\#E(\mathbf{F}_q)$      **ellcard(E)**  
 $\#E(\mathbf{F}_q)$  with almost prime order     **ellsea(E, \{tors\})**  
structure  $\mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z}$  of  $E(\mathbf{F}_q)$      **ellgroup(E)**  
is  $E$  supersingular?     **ellissupersingular(E)**  
Weil pairing of  $m$ -torsion pts  $P, Q$      **ellweilpairing(E, P, Q, m)**  
Tate pairing of  $P, Q$ ;  $P$   $m$ -torsion     **elltatepairing(E, P, Q, m)**  
Discrete log, find  $n$  s.t.  $P = [n]Q$      **elllog(E, P, Q, \{ord\})**

## Curves over Q

**Reduction, minimal model**  
minimal model of  $E/\mathbf{Q}$      **ellminimalmodel(E, \{\&v\})**  
quadratic twist of minimal conductor     **ellminimaltwist(E)**  
 $[k]P$  with good reduction     **ellnonsingularmultiple(E, P)**  
 $E$  supersingular at  $p$ ?     **ellissupersingular(E, p)**  
affine points of naïve height  $\leq h$      **ellratpoints(E, h)**

**Complex heights**  
canonical height of  $P$      **ellheight(E, P)**  
canonical bilinear form taken at  $P, Q$      **ellheight(E, P, Q)**  
height regulator matrix for pts in  $L$      **ellheightmatrix(E, L)**

**p-adic heights**  
cyclotomic  $p$ -adic height of  $P \in E(\mathbf{Q})$      **ellpadicheight(E, p, n, P)**  
... bilinear form at  $P, Q \in E(\mathbf{Q})$      **ellpadicheight(E, p, n, P, Q)**  
... matrix at vector for pts in  $L$      **ellpadicheightmatrix(E, p, n, L)**  
... regulator for canonical height     **ellpadicregulator(E, p, n, Q)**  
Frobenius on  $\mathbf{Q}_p \otimes H_{dR}^1(E/\mathbf{Q})$      **ellpadicfrobienus(E, p, n)**  
slope of unit eigenvector of Frobenius     **ellpads2(E, p, n)**

**Isogenous curves**  
matrix of isogeny degrees for **Q**-isog. curves     **ellisomat(E)**  
tree of prime degree isogenies     **ellisotree(E)**  
a modular equation of prime degree  $N$      **ellmodulareqn(N)**

**L-function**  
 $p$ -th coeff  $a_p$  of  $L$ -function,  $p$  prime     **ellap(E, p)**  
 $k$ -th coeff  $a_k$  of  $L$ -function     **ellak(E, k)**  
 $L(E, s)$  (using less memory than **lfun**)     **elllseries(E, s)**  
 $L^{(r)}(E, 1)$  (using less memory than **lfun**)     **elll1(E, r)**  
a Heegner point on  $E$  of rank 1     **ellheegner(E)**  
order of vanishing at 1     **ellanalyticrank(E, \{eps\})**  
root number for  $L(E, \cdot)$  at  $p$      **ellrootno(E, \{p\})**  
modular parametrization of  $E$      **elltaniyama(E)**  
degree of modular parametrization     **ellmoddegree(E)**  
compare with  $H^1(X_0(N), \mathbf{Z})$  (for  $E' \rightarrow E$ )     **ellweilcurve(E)**

$p$ -adic  $L$  function  $L_p^{(r)}(E, d, \chi^s)$      **ellpadicL(E, p, n, \{s\}, \{r\}, \{d\})**  
BSD conjecture for  $L_p^{(r)}(E_D, \chi^0)$      **ellpadicbsd(E, p, n, \{D = 1\})**  
Iwasawa invariants for  $L_p(E_D, \tau^i)$      **ellpadiclambda(E, p, D, i)**

**Elldata package, Cremona's database:**  
db code "11a1"  $\leftrightarrow$  [*conductor, class, index*]     **ellconvertname(s)**  
generators of Mordell-Weil group     **ellgenerators(E)**  
look up  $E$  in database     **ellidentify(E)**  
all curves matching criterion     **ellsearch(N)**  
loop over curves with cond. from  $a$  to  $b$      **forell(E, a, b, seq)**

## Curves over number field $K$

coeff  $a_p$  of  $L$ -function     **ellap(E, p)**  
Kodaira type of  $\mathfrak{p}$ -fiber of  $E$      **elllocalred(E, p)**  
integral model of  $E/K$      **ellintegralmodel(E, \{\&v\})**  
minimal model of  $E/K$      **ellminimalmodel(E, \{\&v\})**  
minimal discriminant of  $E/K$      **ellminimaldisc(E)**  
cond, min mod, Tamagawa num  $[N, v, c]$      **ellglobalred(E)**  
global Tamagawa number     **elltamagawa(E)**  
 $P \in E(K)$   $n$ -divisible?  $[n]Q = P$      **ellisdivisible(E, P, n, \{\&Q\})**

**L-function**  
A domain  $D = [c, w, h]$  in initialization mean we restrict  $s \in \mathbf{C}$  to domain  $|\Re(s) - c| < w, |\Im(s)| < h; D = [w, h]$  encodes  $[1/2, w, h]$  and  $[h]$  encodes  $D = [1/2, 0, h]$  (critical line up to height  $h$ ).  
vector of first  $n$   $a_k$ 's in  $L$ -function     **ellan(E, n)**  
init  $L^{(k)}(E, s)$  for  $k \leq n$      **L = lfunit(E, D, \{n = 0\})**  
compute  $L(E, s)$  ( $n$ -th derivative)     **lfun(L, s, \{n = 0\})**  
 $L(E, 1, r)/(r! \cdot R \cdot \#Sha)$  assuming BSD     **ellbsd(E)**

## Other curves of small genus

A hyperelliptic curve is given by a pair  $[P, Q]$  ( $y^2 + Qy = P$  with  $Q^2 + 4P$  squarefree) or a single squarefree polynomial  $P$  ( $y^2 = P$ ).  
reduction of  $y^2 + Qy = P$  (genus 2)     **genus2red([P, Q], \{p\})**  
affine rational points of height  $\leq h$      **hyperellratpoints([P, Q], h)**  
find a rational point on a conic,  ${}^t xGx = 0$      **qfsolve(G)**  
quadratic Hilbert symbol (at  $p$ )     **hilbert(x, y, \{p\})**  
all solutions in  $\mathbf{Q}^3$  of ternary form     **qfparam(G, x)**  
 $P, Q \in \mathbf{F}_q[X]$ ; char. poly. of Frobenius     **hyperellcharpoly([P, Q])**  
matrix of Frobenius on  $\mathbf{Q}_p \otimes H_{dR}^1$      **hyperellpadicfrobienus**

## Elliptic & Modular Functions

$w = [\omega_1, \omega_2]$  or *ell* struct (**E.omega**),  $\tau = \omega_1/\omega_2$ .  
arithmetic-geometric mean     **agm(x, y)**  
elliptic  $j$ -function  $1/q + 744 + \dots$      **ellj(x)**  
Weierstrass  $\sigma/\wp/\zeta$  function     **ellsigma(w, z), ellwp, ellzeta**  
periods/quasi-periods     **ellperiods(E, \{flag\}), elleta(w)**  
 $(2i\pi/\omega_2)^k E_k(\tau)$      **elleisnum(w, k, \{flag\})**  
modified Dedekind  $\eta$  func.  $\prod(1 - q^n)$      **eta(x, \{flag\})**  
Dedekind sum  $s(h, k)$      **sumdedekind(h, k)**  
Jacobi sine theta function     **theta(q, z)**  
 $k$ -th derivative at  $z=0$  of **theta(q, z)**     **thetanullk(q, k)**  
Weber's  $f$  functions     **weber(x, \{flag\})**  
modular pol. of level  $N$      **polmodular(N, \{inv = j\})**  
Hilbert class polynomial for  $\mathbf{Q}(\sqrt{D})$      **polclass(D, \{inv = j\})**

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