Poincaré à la Makdisi

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2026 PARI/GP workshop
13 January 2026

The Poincaré torsor

Line bundles and torsors

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The fibres of $\mathcal{L}^\times$ are torsors under $\mathbb{G}_m$ (free and transitive action of $K^\times$).

Enters Poincaré

Now let A be an Abelian variety over K.
Its dual is

$$A^\vee = \mathrm{Pic}^0(A) = \{\text{line bundles } \mathcal{L} \to A \mid \deg \mathcal{L} = 0\}/\simeq .$$

The Poincaré bundle on A is the unique line bundle

$$\mathcal{P} \longrightarrow A \times A^\vee$$

such that

- for all $y = [\mathcal{L}] \in A^\vee$, $\mathcal{P}_{|A \times \{y\}} \simeq \mathcal{L}$,
- and $\mathcal{P}_{|0 \times A^\vee} \simeq \mathcal{O}_{A^\vee}$ is trivial.

The Poincaré torsor on A is $\mathcal{P}^\times$.

What this means for Jacobians

Let C be a "nice" curve over K.
Its Jacobian is

$$\begin{aligned} J &\overset{\text{def}}{=} \operatorname{Pic}^0(C) \\ &\overset{\text{def}}{=} \{\text{line bundles } \mathcal{L} \to C \mid \deg \mathcal{L} = 0\}/\simeq \\ &\underset{\mathcal{O}_C(D) \hookleftarrow D}{=} \{\text{divisors } D/C \mid \deg D = 0\}/\sim . \end{aligned}$$

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Theorem

Let $x = [D]$ and $y = [E] \in J$.
The stalk of $\mathcal{P}$ at $(x, y) \in J \times J$ is

$$\mathcal{P}_{x,y} \simeq \mathcal{N}_D(\mathcal{O}_C(E)).$$

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Here $\mathcal{N}_D(f) \stackrel{\text{def}}{=} f(D) \stackrel{\text{def}}{=} \prod_i f(P_i)^{n_i}$ where $D = \sum_i n_i P_i$.

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Thus if $D \not\cap E$, then the meromorphic section 1 of $\mathcal{O}_C(E)$ is regular and nonvanishing along D, so

$$\mathcal{N}_D\big(1 \in \mathcal{O}_C(E)\big) \in \mathcal{P}_{x,y}^\times,$$

and

$$\mathcal{P}_{x,y}^\times = \Big\{\lambda \cdot \mathcal{N}_D\big(1 \in \mathcal{O}_C(E)\big) \;\Big|\; \lambda \in K^\times\Big\}.$$

Partial group laws

We have partial group laws

$$\mathcal{P}^{\times}_{x_1,y} \otimes \mathcal{P}^{\times}_{x_2,y} \longrightarrow \mathcal{P}^{\times}_{x_1+x_2,y} \quad \text{and} \quad \mathcal{P}^{\times}_{x,y_1} \otimes \mathcal{P}^{\times}_{x,y_2} \longrightarrow \mathcal{P}^{\times}_{x,y_1+y_2}$$

coming from

$$\mathcal{N}_{D_1}\big(\mathcal{O}_C(E)\big) \otimes \mathcal{N}_{D_2}\big(\mathcal{O}_C(E)\big) \simeq \mathcal{N}_{D_1+D_2}\big(\mathcal{O}_C(E)\big),$$

$$\mathcal{N}_{D}\big(\mathcal{O}_C(E_1)\big) \otimes \mathcal{N}_{D}\big(\mathcal{O}_C(E_2)\big) \simeq \mathcal{N}_{D}\big(\mathcal{O}_C(E_1+E_2)\big).$$

Application: Quadratic Chabauty

Classical Chabauty

Let C be a "nice" curve of genus g over $\mathbb{Q}$.

Theorem (Faltings)

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Theorem (Mordell-Weil)

$$J(\mathbb{Q}) \simeq \mathbb{Z}^r \times \textit{finite group.}$$

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Let C be a "nice" curve of genus g over $\mathbb{Q}$.
Suppose we know $P_0 \in C(\mathbb{Q})$. Then

$$j : \begin{array}{ccc} C & \longrightarrow & J \\ P & \longmapsto & [P - P_0] \end{array},$$

is an embedding (assuming $g \neq 0$).

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Idea (Chabauty)

Fix a <u>Chabauty prime</u> $p \in \mathbb{N}$; catch $C(\mathbb{Q})$ inside

$$\underbrace{C(\mathbb{Q}_p)}_{\dim 1} \cap \underbrace{\overline{J(\mathbb{Q})}^{p\text{-adic}}}_{\dim \leqslant r} \subset \underbrace{J(\mathbb{Q}_p)}_{\dim g}.$$

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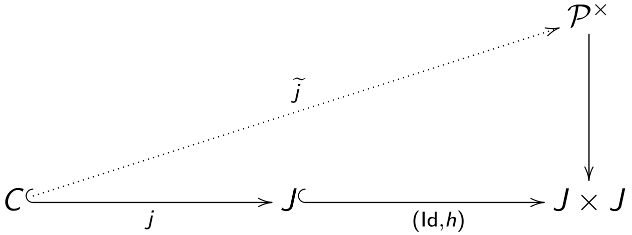
$$\underbrace{C(\mathbb{Q}_p)}_{\dim 1} \cap \underbrace{\overline{J(\mathbb{Q})}^{p\text{-adic}}}_{\dim \leqslant r} \subset \underbrace{J(\mathbb{Q}_p)}_{\dim g}.$$

But what if $r \geqslant g$? Then $J(\mathbb{Q})$ could be dense in $J(\mathbb{Q}_p)$ ...

Quadratic Chabauty

Try to gain a dimension by lifting $j : \begin{array}{ccc} C & \hookrightarrow & J \\ P & \mapsto & [P - P_0] \end{array}$ to $\mathcal{P}^\times$:



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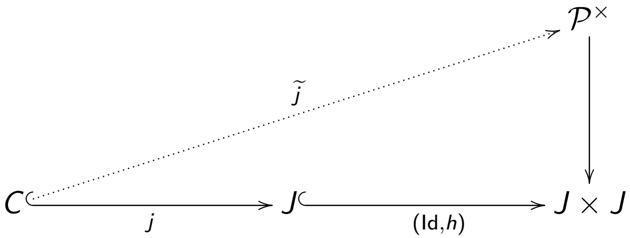

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We now try to catch $C(\mathbb{Q})$ inside

$$C(\mathbb{Q}_p) \cap \underbrace{\overline{\mathcal{P}^\times(\mathbb{Q})}^{p\text{-adic}}} \subset \mathcal{P}^\times(\mathbb{Q}_p).$$

Unfortunately, $\mathbb{Q}^\times$ dense in $\mathbb{Q}_p^\times$, so $\mathcal{P}^\times(\mathbb{Q})$ dense in $\mathcal{P}^\times(\mathbb{Q}_p)$ whenever $J(\mathbb{Q})$ dense in $J(\mathbb{Q}_p)$ ...

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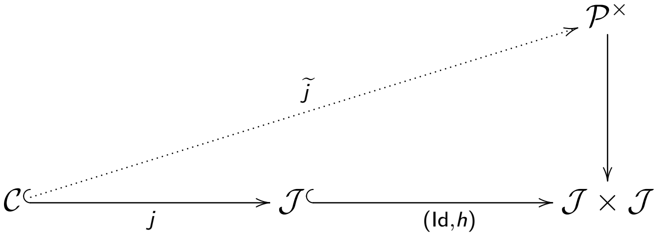
Try to gain a dimension by lifting $j : \begin{array}{ccc} C & \hookrightarrow & J \\ P & \mapsto & [P - P_0] \end{array}$ to $\mathcal{P}^\times$:

Let $\mathcal{C}/\mathbb{Z}$ be a proper model of $C/\mathbb{Q}$, so that $\mathcal{C}(\mathbb{Z}) = C(\mathbb{Q})$, and let $\mathcal{J}$ be the Néron model of J:



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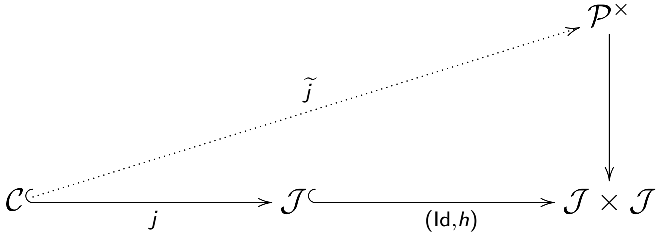

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Since $\mathbb{G}_m(\mathbb{Z}) = \mathbb{Z}^\times = \{\pm 1\}$ is finite, we know the possible $\mathbb{G}_m$-components of $\widetilde{j}(\mathcal{C}(\mathbb{Z}))$.

Implementing J and $\mathcal{P}^\times$

Makdisi's framework

Let C/K be a "nice" curve of genus g.
Assume $C(K)$ large (e.g. $K = \mathbb{C}$ or $\mathbb{Q}_p$, or extend K).

Pick line bundle $\mathcal{L} \to C$ of degree $d_0 = \deg \mathcal{L} \gg_g 0$,
and fix points $Q_1, \cdots, Q_m \in C(K)$ $(m \gg_{d_0} 0)$.

Write $V_n =$ global sections of $\mathcal{L}^{\otimes n}$ $(n = 1, 2, \cdots, 5)$.
Each $0 \neq s \in V_n$ has divisor $(s)_n$, effective of degree $n \cdot d_0$.

More generally, when D is an effective divisor, write
$V_n(-D) \subset V_n$ for the sections of $\mathcal{L}^{\otimes n}(-D)$.

Embed the V_n into K^m by $s \mapsto \big(s(Q_1), \cdots, s(Q_m)\big)$.

Each point $x \in J$ is of the form $x = [\mathcal{L}(-D)]$ for some
effective divisor D of degree d_0, and is represented by

$$V_2(-D) \stackrel{\text{def}}{=} \text{ sections of } \mathcal{L}^{\otimes 2}(-D) \subset V_2 \subset K^m.$$

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In particular, $0 \in J$ is represented by $s \cdot V_1 \subset V_2$ for any $0 \neq s \in V_1$, corresponding to $D = (s)_1$.

Equality test

Algorithm (Makdisi's equality test)

Given $V_2(-D)$ and $V_2(-D')$
representing $x = [\mathcal{L}(-D)]$ and $y = [\mathcal{L}(-D')] \in J$,

1. Pick $0 \neq u \in V_2(-D) \rightsquigarrow (u)_2 = D + E$.
2. Compute $u \cdot V_2(-D') = V_4(-D - D' - E)$.
3. Compute
   $W = V_2(-D' - E) = \{v \in V_2 \mid v \cdot V_2(-D) \subset u \cdot V_2(-D')\}$.

- If $x = y$, then $W = K \cdot u'$ where $D + (u')_2 = D' + (u)_2$.
- If $x \neq y$, then $W = \{0\}$.

Note: the "linear equivalence certificate" $u/u' \in K(C)$ is randomised.

Representing $\mathcal{P}^{\times}_{x,y}$

Let $x = [\mathcal{L}(-D)]$ and $y = [\mathcal{L}(-E)] \in J$. What is $\mathcal{P}^{\times}_{x,y}$?

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Fix $s_0 \in V_1$, and let $D_0 \overset{\text{def}}{=} (s_0)_1$.
Then $\mathcal{L} \simeq \mathcal{O}_C(D_0) \rightsquigarrow x = [D - D_0]$, $y = [E - D_0]$.

Unfortunately, $D - D_0$ and $E - D_0$ intersect,
so $\mathcal{N}_{D-D_0}\big(1 \in \mathcal{O}_C(E - D_0)\big)$ is <u>not</u> a valid element of $\mathcal{P}^\times_{x,y}$.

Representing $\mathcal{P}_{x,y}^{\times}$

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Solution: fix <u>another</u> $t_0 \in V_1$, such that $E_0 \stackrel{\text{def}}{=} (t_0)_1 \not\cap D_0$.

If $D \not\cap E_0$, $E \not\cap D_0$, and $D \not\cap E$, then

$$[D, E] \stackrel{\text{def}}{=} \mathcal{N}_{D-D_0}\big(1 \in \mathcal{O}_C(E - E_0)\big) \in \mathcal{P}_{x,y}^{\times},$$

and $\mathcal{P}_{x,y}^{\times} = \{\lambda \cdot [D, E] \mid \lambda \in K^{\times}\}$.

$\rightsquigarrow$ We can use $[D, E]$ as a <u>reference point</u> for $\mathcal{P}_{x,y}^{\times}$.

Representing $\mathcal{P}^\times_{x,y}$

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$\rightsquigarrow$ We can use $[D, E]$ as a <u>reference point</u> for $\mathcal{P}^\times_{x,y}$.

However, it depends on the choice of effective divisors D, E such that $x = [\mathcal{L}(-D)]$ and $y = [\mathcal{L}(-E)]$ ...

Comparison formula

Algorithm (Comparison in $\mathcal{P}^\times$)

Given 4 points $x_1 = [\mathcal{L}(-D_1)]$, $y_1 = [\mathcal{L}(-E_1)]$,
$x_2 = [\mathcal{L}(-D_2)]$, $y_2 = [\mathcal{L}(-E_1)]$ of J,
we want to compare $[D_1, E_1]$ with $[D_2, E_2]$.

1. Use Makdisi's equality test to check $x_1 = x_2$ and $y_1 = y_2$.
   If not, complain / terminate.
   Otherwise, we get $d_1, d_2, e_1, e_2 \in V_2$ such that
   $$D_1 + (d_1)_2 = D_2 + (d_2)_2 \quad \text{and} \quad E_1 + (e_1)_2 = E_2 + (e_2)_2.$$
2. Output that $[D_2, E_2] = \lambda \cdot [D_1, E_1]$, where
   $$\lambda = \frac{\mathcal{N}_{E_2}(d_1/d_2)\ \mathcal{N}_{D_0}(e_1/e_2)}{\mathcal{N}_{E_0}(d_1/d_2)\ \mathcal{N}_{D_1}(e_1/e_2)} = \frac{\mathcal{N}_{E_1}(d_1/d_2)\ \mathcal{N}_{D_0}(e_1/e_2)}{\mathcal{N}_{E_0}(d_1/d_2)\ \mathcal{N}_{D_2}(e_1/e_2)}.$$

Notes: $V_2(-D_0) = s_0 \cdot V_1$, and $V_2(-E_0) = t_0 \cdot V_1$.
May need to run several times until the norms work.

Evaluating norms

Given $V_2(-D)$ encoding effective $D = \sum_i n_i P_i$, and columns in K^m representing $u, v \in V_2$ not vanishing at the P_i, want to evaluate

$$\mathcal{N}_D(u/v) = \prod_i \frac{u}{v}(P_i)^{n_i} \in K^\times.$$

Algorithm (Norm)

1. Compute $V_4(-D) = V_2 \cdot V_2(-D)$.
2. Find supplements $V_2 = S \oplus V_2(-D)$ and $V_4 = T \oplus V_4(-D)$.
3. Compute $\Delta_u = \det\left(S \hookrightarrow V_2 \xrightarrow{\cdot u} V_4 \twoheadrightarrow T \right)$ and Δ_v.
4. Output $\mathcal{N}_D(u/v) = \Delta_u / \Delta_v$.

Explanation: $V(-D) = \operatorname{Ker}\left(s \mapsto s(D)\right)$.

Summary

- We fix once and for all $\mathcal{L} \gg 0 \longrightarrow C$,
and then two sections s_0, t_0 of $\mathcal{L}$ such that
$$D_0 \stackrel{\text{def}}{=} (s_0)_1 \quad \not\cap \quad E_0 \stackrel{\text{def}}{=} (t_0)_1.$$
- Given $x, y \in J$, choose effective divisors D, E such that
$$x = [\mathcal{L}(-D)],\ y = [\mathcal{L}(-E)] \quad \text{and} \quad D \not\cap E_0,\ E \not\cap D_0,\ D \not\cap E,$$
and represent them by
$$V_2(-D),\ V_2(-E) \quad \subset V_2 = \text{sections of } \mathcal{L}^{\otimes 2}.$$
- Then $[D, E] \stackrel{\text{def}}{=} \mathcal{N}_{D-D_0}\big(1 \in \mathcal{O}_C(E - E_0)\big) \in \mathcal{P}^\times_{x,y}$,
so we represent $\lambda \cdot [D, E] \in \mathcal{P}^\times_{x,y}$ by the triple
$$\Big(\quad V_2(-D) \quad , \quad V_2(-E) \quad , \quad \lambda \quad \Big)$$
(where $\lambda \in K^\times$).

Group law in J

Algorithm (Makdisi's addflip)

Given $V_2(-D_1)$ and $V_2(-D_2)$
representing $x_1 = [\mathcal{L}(-D_1)]$ and $x_2 = [\mathcal{L}(-D_2)] \in J$,

compute $V_2(-D_3)$ representing $x_3 = [\mathcal{L}(-D_3)]$
such that $x_1 + x_2 + x_3 = 0 \in J$.

1. Compute $V_4(-D_1 - D_2) = V_2(-D_1) \cdot V_2(-D_2)$.
2. Compute
   $V_3(-D_1 - D_2) = \{v \in V_3 \mid v \cdot V_1 \subset V_4(-D_1 - D_2)\}$.
3. Pick $0 \neq u \in V_3(-D_1 - D_2)$, so $(u)_3 = D_1 + D_2 + D_3$.
4. Compute $u \cdot V_2 = V_5(-D_1 - D_2 - D_3)$.
5. Compute
   $V_2(-D_3) = \{v \in V_2 \mid v \cdot V_3(-D_1 - D_2) \subset u \cdot V_2\}$.

Group law in J

Algorithm (Makdisi's negation)

Given $V_2(-D)$ representing $x = [\mathcal{L}(-D)] \in J$,
compute $V_2(-D')$ representing $x' = [\mathcal{L}(-D')] \in J$
such that $x + x' = 0 \in J$.

1. Pick $0 \neq u \in V_2(-D)$, so that $(u)_2 = D + D'$.
2. Compute $u \cdot V_2 = V_4(-D - D')$.
3. Compute $V_2(-D') = \{v \in V_2 \mid v \cdot V_2(-D) \subset u \cdot V_2\}$.

Partial group laws in $\mathcal{P}^\times$

Algorithm (Left partial group law in $\mathcal{P}^\times$)

Given 4 points of J
$x_1 = [\mathcal{L}(-D_1)]$, $y_1 = [\mathcal{L}(-E_1)]$, $x_2 = [\mathcal{L}(-D_2)]$, $y_2 = [\mathcal{L}(-E_2)]$
such that $y_1 = y_2 \overset{\text{def}}{=} y$,

want to apply $\mathcal{P}^\times_{x_1,y} \otimes \mathcal{P}^\times_{x_2,y} \to \mathcal{P}^\times_{x_1+x_2,y}$ to $[D_1, E_1]$ and $[D_2, E_2]$.

1. Find $\lambda \in K^\times$ such that $[D_2, E_2] = \lambda [D_2, E_1]$.
2. Find $(V_2(-D_3), u)$ such that $(u)_3 = D_1 + D_2 + D_3$, and then $(V_2(-D_4), v)$ such that $(v)_2 = D_3 + D_4$.
3. Output $\lambda \cdot \mu \cdot [D_4, E_1]$, where $\mu = \mathcal{N}_{E_1}\left(\frac{u}{vs_0}\right) \Big/ \mathcal{N}_{E_0}\left(\frac{u}{vs_0}\right)$.

The right partial group law is similar.