Algebraic number theory with GP

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Documentation

- ▶ refcard-nf.pdf: list of functions with a short description.
- ▶ users.pdf Section 3.13: introduction and detailed descriptions of the functions.
- ▶ in gp, `?10`: list of functions.
- ▶ in gp, `?functionname`: short description of the function.
- ▶ in gp, `??functionname`: long description of the function.

To record the commands we will type during the tutorial:

```
? \l TAN.log
```

Plan

Number fields

Ideals

Class groups and units

Number Fields

Irreducibility

In GP, we describe a number field *K* as

$$K = \mathbb{Q}[x]/f(x)$$

where $f \in \mathbb{Z}[x]$ is a monic irreducible polynomial.

```
? f = x^4 - 2*x^3 + x^2 - 5;
? polisirreducible(f)
% = 1
```

GP knows cyclotomic polynomials:

```
? g = polcyclo(30)
% = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1
```

Algebraic numbers

To perform simple operations in $K = \mathbb{Q}[x]/f(x) = \mathbb{Q}(\alpha)$ where $f(\alpha) = 0$, we can use Mod:

```
? Mod(x,f)^5
% = Mod(3*x^3-2*x^2+5*x+10, x^4-2*x^3+x^2-5)
```

Interpretation: $\alpha^5 = 3\alpha^3 - 2\alpha^2 + 5\alpha + 10$.

We check that the roots of *g* are 30th roots of unity:

```
? lift(Mod(x,g)^15)
% = -1
```

We used lift to make the output more readable.

polredbest

Sometimes we can find a simpler defining polynomial for the same number field by using `polredbest`:

```
? {h = x^5 + 7*x^4 + 22550*x^3 - 281686*x^2
  - 85911*x + 3821551};
? polredbest(h)
% = x^5 - x^3 - 2*x^2 + 1
```

Interpretation: $\mathbb{Q}[x]/h(x) \cong \mathbb{Q}[x]/(x^5 - x^3 - 2x^2 + 1)$.

nfinit

Most operations on number fields use a structure representing the field and its ring of integers, which is computed by `nfinit`.

```
? K = nfinit(f);
```

K contains the structure for the number field $K = \mathbb{Q}[x]/f(x)$.

```
? K.pol
% = x^4 - 2*x^3 + x^2 - 5
? K.sign
% = [2, 1]
```

K has signature $(2, 1)$: it has two real embeddings and one pair of conjugate complex embeddings.

Computed information

```
? K.disc
% = -1975
? K.zk
% = [1,1/2*x^2-1/2*x-1/2,x,1/2*x^3-1/2*x^2-1/2*x]
? w = K.zk[2];
```

K has discriminant -1975, and its ring of integers is

$$\mathbb{Z}_K = \mathbb{Z} + \mathbb{Z}\frac{\alpha^2 - \alpha - 1}{2} + \mathbb{Z}\alpha + \mathbb{Z}\frac{\alpha^3 - \alpha^2 - \alpha}{2} = \mathbb{Z} + \mathbb{Z}w + \mathbb{Z}\alpha + \mathbb{Z}w\alpha.$$

Elements of a number field

We saw that we could represent elements of a number field as polynomials in α. We can also use linear combinations of the integral basis. We can switch between the two representations with nfalgtobasis and nfbasistoalg.

```
? nfalgtobasis(K,x^2)
% = [1, 2, 1, 0]~
```

Interpretation: $\alpha^2 = 1 \cdot 1 + 2 \cdot w + 1 \cdot \alpha + 0 \cdot w\alpha = 1 + 2w + \alpha$.

```
? nfbasistoalg(K,[1,1,1,1]~)
% = Mod(1/2*x^3 + 1/2, x^4 - 2*x^3 + x^2 - 5)
```

Interpretation: $1 + w + \alpha + w\alpha = \frac{\alpha^3+1}{2}$.

Elements of a number field: operations

We perform operations on elements with the functions
`nfeltxxxx`, which accept both representations as input.

```
? nfeltmul(K,[1,-1,0,0]~,x^2)
% = [-1, 3, 1, -1]~
```

Interpretation: $(1 - w) \cdot \alpha^2 = -1 + 3w + \alpha - w\alpha$.

```
? nfeltnorm(K,x-2)
% = -1
? nfelttrace(K,[0,1,2,0]~)
% = 2
```

Interpretation: $N_{K/\mathbb{Q}}(\alpha - 2) = -1$, $\mathrm{Tr}_{K/\mathbb{Q}}(w + 2\alpha) = 2$.

Ideals

Reminder

In $\mathbb{Z}_K$, ideals factor uniquely into products of prime ideals:

$$\mathfrak{a} = \prod_i \mathfrak{p}_i^{a_i}.$$

In particular, prime numbers admit a decomposition:

$$p\mathbb{Z}_K = \prod_i \mathfrak{p}_i^{e_i} \text{ with } \mathbb{Z}_K/\mathfrak{p}_i \cong \mathbb{F}_{p^{f_i}}.$$

▶ e_i = ramification index of $\mathfrak{p}_i$.

▶ f_i = residue degree of $\mathfrak{p}_i$.

Decomposition of primes

We can decompose primes with idealprimedec:

```
? dec = idealprimedec(K,5);
? #dec
% = 2
? [pr1,pr2] = dec;
```

Interpretation: $\mathbb{Z}_K$ has two prime ideals above 5, which we call $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

```
? pr1.f
% = 1
? pr1.e
% = 2
```

$\mathfrak{p}_1$ has residue degree 1 and ramification index 2.

Decomposition of primes

```
? pr1.gen
% = [5, [-1, 0, 1, 0]~]
```

$\mathfrak{p}_1$ is generated by 5 and $-1 + 0 \cdot w + \alpha + 0 \cdot w\alpha$, i.e. we have $\mathfrak{p}_1 = 5\mathbb{Z}_K + (\alpha - 1)\mathbb{Z}_K$.

```
? pr2.f
% = 1
? pr2.e
% = 2
```

$\mathfrak{p}_2$ also has residue degree 1 and ramification index 2.

Ideals

An arbitrary ideal is represented by its Hermite normal form (HNF) with respect to the integral basis. We can obtain this form with `idealhnf`.

```
? idealhnf(K,pr1)
% =
[5 3 4 3]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

Interpretation: $\mathfrak{p}_1$ can be described as

$$\mathfrak{p}_1 = \mathbb{Z} \cdot 5 + \mathbb{Z} \cdot (w + 3) + \mathbb{Z} \cdot (\alpha + 4) + \mathbb{Z} \cdot (w\alpha + 3).$$

Ideals

```
? a = idealhnf(K,[23, 10, -5, 1]~)
% =
[260   0 228 123]
[  0 260 123 105]
[  0   0   1   0]
[  0   0   0   1]
```

We obtain the HNF of the ideal $\mathfrak{a} = (23 + 10w - 5\alpha + w\alpha)$.

```
? idealnorm(K,a)
% = 67600
```

We have $N(\mathfrak{a}) = 67600$.

Ideals: operations

We perform operations on ideals with the functions
`idealxxxx`, which accept HNF forms, prime ideal structures
(output of `idealprimedec`), and elements (interpreted as
principal ideals).

```
? idealpow(K,pr2,3)
% =
[25 15 21 7]
[ 0  5  2 4]
[ 0  0  1 0]
[ 0  0  0 1]
? idealnorm(K,idealadd(K,a,pr2))
% = 1
```

We have $\mathfrak{a} + \mathfrak{p}_2 = \mathbb{Z}_K$: the ideals $\mathfrak{a}$ and $\mathfrak{p}_2$ are coprime.

Ideals: factorisation

We factor an ideal into a product of prime ideals with idealfactor. The result is a two-column matrix: the first column contains the prime ideals, and the second one contains the exponents.

```
? fa = idealfactor(K,a);
? matsize(fa)
% = [3,2]
```

The ideal a is divisible by three prime ideals.

```
? [fa[1,1].p, fa[1,1].f, fa[1,1].e, fa[1,2]]
% = [2, 2, 1, 2]
```

The first one is a prime ideal above 2, is unramified with residue degree 2, and appears with exponent 2.

Ideals: factorisation

```
? [fa[2,1].p, fa[2,1].f, fa[2,1].e, fa[2,2]]
% = [5, 1, 2, 2]
? fa[2,1]==pr1
% = 1
```

The second one is $\mathfrak{p}_1$, and it appears with exponent 2.

```
? [fa[3,1].p, fa[3,1].f, fa[3,1].e, fa[3,2]]
% = [13, 2, 1, 1]
```

The third one is a prime ideal above 13, is unramified with residue degree 2, and appears with exponent 2.

Chinese remainders

We can use the Chinese remainder theorem with
`idealchinese`:

```
? b = idealchinese(K,[pr1,2;pr2,1],[1,-1]);
```

We are looking for an element $b \in \mathbb{Z}_K$ such that $b = 1 \bmod \mathfrak{p}_1^2$
and $b = -1 \bmod \mathfrak{p}_2$.

```
? nfeltval(K,b-1,pr1)
% = 2
? nfeltval(K,b+1,pr2)
% = 1
```

We check the output by computing valuations: $v_{\mathfrak{p}_1}(b - 1) = 2$
and $v_{\mathfrak{p}_2}(b + 1) = 1$.

Chinese remainders with signs

We can compute the sign of real embeddings of *b*:

```
? nfeltsign(K,b)
% = [-1, 1]
```

We have $\sigma_1(b) < 0$ and $\sigma_2(b) > 0$, where σ_1, σ_2 are the two real embeddings of *K*.

We can ask idealchinese to compute an element that, in addition to the congruences, is totally positive:

```
? c = idealchinese(K, [[pr1,2;pr2,1],[1,1]],[1,-1]);
? nfeltsign(K,c)
% = [1, 1]
```

Indeed we have $\sigma_1(c) > 0$ and $\sigma_2(c) > 0$.

Reminder

The Dedekind zeta function is defined for $s \in \mathbb{C}$ with $\Re(s) > 1$:

$$\zeta_K(s) = \sum_{\mathfrak{a}} N(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} \frac{1}{1 - N(\mathfrak{p})^{-s}}$$

where the sum ranges over nonzero ideals $\mathfrak{a}$ of $\mathbb{Z}_K$, and the product over prime ideals $\mathfrak{p}$ of $\mathbb{Z}_K$.

ζ_K extends to a meromorphic function on $\mathbb{C}$, with a simple pole at $s = 1$ and a functional equation relating $\zeta_K(s)$ and $\zeta_K(1 - s)$.

Dedekind zeta function

We can evaluate the Dedekind zeta function with `lfun`.

```
? F = nfinit(x^3-3*x-1);
? F.sign
% = [3, 0]
```

F is totally real.

```
? lfun(F,2)
% = 1.1722471496117109428809260096356285918
? q = bestappr(lfun(F,2)/Pi^6)
% = 8/6561
? lfun(F,2)/(Pi^6*q)
% = 1.0000000000000000000000000000000000000
```

$\zeta_F(2)$ is a rational multiple of π^6 (Siegel's theorem).

Class groups and units

Reminder

The class group

$$\mathrm{Cl}(K) = \frac{(\text{nonzero ideals of } K)}{(\text{principal ideals } \beta \mathbb{Z}_K)}.$$

is a finite abelian group.

The unit group

$$\mathbb{Z}_K^\times \cong \mathbb{Z}/w\mathbb{Z} \times \mathbb{Z}^{r_1+r_2-1}$$

is a lattice under the logarithmic embedding, whose covolume is called the regulator Reg_K.

bnfinit

To obtain the class group and unit group of a number field, we need a more expensive computation than `nfinit`. The relevant information is contained in the structure computed with `bnfinit` (b = Buchmann).

```
? K2 = bnfinit(K);
? K2.nf == K
% = 1
? K2.no
% = 1
```

K has a trivial class group (no = class number).

```
? K2.reg
% = 1.7763300299706546701307646106399605586
```

We obtain an approximation of the regulator of *K*.

bnfcertify

The output of `bnfisprincipal` is a priori only correct under GRH (Generalised Riemann Hypothesis). We can unconditionally certify it with `bnfcertify`.

```
? bnfcertify(K2)
% = 1
```

The computation is now certified! If `bnfcertify` outputs `0`, it means we have found a counter-example to GRH (or more likely a bug in PARI/GP)!

bnfinit: units

```
? lift(K2.tu)
% = [2, -1]
? K2.tu[1]==nfrootsof1(K)[1]
% = 1
```

K has two roots of unity (tu = torsion units), ± 1. We can also compute them with `nfrootsof1`.

```
? lift(K2.fu)
% = [1/2*x^2-1/2*x-1/2, 1/2*x^3-3/2*x^2+3/2*x-1]
```

The free part of $\mathbb{Z}_K^\times$ is generated by $\frac{\alpha^2-\alpha-1}{2}$ and $\frac{\alpha^3-3\alpha^2+3\alpha-2}{2}$ (fu = fundamental units).

bnfinit: analytic class number formula

```
? lfun(K,1+x+O(x^2))
% = 0.50228472605280111386617636567964565 51*x^-1
  + O(x^0)
? res = polcoeff(lfun(K,1+x+O(x^2)),-1)
% = 0.50228472605280111386617636567964565169
```

We compute an approximation of the residue of $\zeta_K(s)$ at $s = 1$.

```
? {2^K2.r1*(2*Pi)^K2.r2*K2.no*K2.reg/
     (K2.tu[1]*sqrt(abs(K2.disc))*res)}
% = 0.99999999999999999999999999999999999999
```

We numerically check the analytic class number formula.

Class group

```
? L = bnfinit(x^3 - x^2 - 54*x + 169);
? L.cyc
% = [2, 2]
```

$Cl(L) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

```
? L.gen
% = [[5,3,2;0,1,0;0,0,1], [5,4,3;0,1,0;0,0,1]]
```

Generators of the class group, given as ideals in HNF form.

Testing whether an ideal is principal

We test whether an ideal is principal with `bnfisprincipal`:

```
? pr = idealprimedec(L,13)[1]
? [dl,g] = bnfisprincipal(L,pr);
? dl
% = [1, 0]~
```

`bnfisprincipal` expresses the class of the ideal in terms of the generators of the class group (discrete logarithm). Here, the ideal `pr` is in the same class as the first generator. In particular, the ideal is not principal, but its square is.

Testing whether an ideal is principal

```
? g
% = [0, 1/5, 2/5]~
? {idealhnf(L,pr) == idealmul(L,g,
    idealfactorback(L,L.gen,dl))}
% = 1
```

The second component of the output of bnfisprincipal is
an element $g \in L$ that generates the remaining principal ideal.
(idealfactorback = inverse of idealfactor = $\prod_i$ L.gen[i]$^{\text{dl[i]}}$)

Computing a generator of principal ideal

We know that pr is a 2-torsion element; let's compute a generator of its square:

```
? [dl2,g2] = bnfisprincipal(L,idealpow(L,pr,2));
? dl2
% = [0, 0]~
```

The ideal is indeed principal (trivial in the class group).

```
? g2
% = [1, -1, -1]~
? idealhnf(L,g2) == idealpow(L,pr,2)
% = 1
```

$g2$ is a generator of pr^2.

Application: bnfisintnorm

We can use these functionalities to find solutions in $\mathbb{Z}_K$ of norm equations with `bnfisintnorm`:

```
? bnfisintnorm(L,5)
% = []
```

There is no element of norm 5 in $\mathbb{Z}_L$.

```
? bnfisintnorm(L,65)
% = [x^2 + 4*x - 36, -x^2 - 3*x + 39, -x + 2]
```

There are three elements of $\mathbb{Z}_L$ of norm 65, up to multiplication by elements of $\mathbb{Z}_L^\times$ with positive norm.

Expressing a unit in terms of the generators

```
? u = [0,2,1]~;
? nfeltnorm(L,u)
% = 1
```

We have found a unit $u \in \mathbb{Z}_L^\times$.

```
? bnfisunit(L,u)
% = [1, 2, 0]~
? lift(L.fu)
% = [x^2 + 4*x - 34, x - 4]
? lift(L.tu)
% = [2, -1]
```

We express it in terms of the generators with bnfisunit:
$u = (\alpha^2 + 4\alpha - 34) \cdot (\alpha - 4)^2 \cdot (-1)^0$.

Large fundamental units

By default, `bnfinit` only computes fundamental units if they are not too large.

```
? M = bnfinit(x^2-3019);
? M.fu
% = 0
```

We can force the computation of fundamental units with `bnfinit(,1)`.

```
? M = bnfinit(x^2-3019,1);
? lift(M.fu)
% = [21389...69789*x - 11752...14330]
```

Note: we can handle very large units with `bnfunits` (advanced use, not described here).

Questions ?

Have fun with GP!