Mordel-Weil rank effective computation by 2-descent

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PARI/GP Workshop January, 2024 Let C be a nice algebraic curve.

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Classic process is to apply Chabauty-Coleman method when $\mathsf{rk}_{\mathbb{Q}}(\mathsf{Jac}(\mathcal{C})) < g(\mathcal{C})$

Theorem (Mordell-Weil)

Let K be a number field, J be an abelian variety over K. There exists $r = \operatorname{rank}_{K}(J) \in \mathbb{N}$, such that

$$J(K) \simeq \underbrace{J_{torsion}(K)}_{finite} \times \mathbb{Z}^r$$

Being given *C* (its equation), how to compute $r = rank_{\mathbb{Q}}(Jac(C))$ in order to check Chabauty-Coleman condition for $K = \mathbb{Q}$?

2-descent algorithm is inspired by the Mordell-Weill theorem's proof, which ends up checking $J(\mathbb{Q})/2J(\mathbb{Q})$ finiteness.

In fact:

$$J(\mathbb{Q})/2J(\mathbb{Q})\simeq J[2](\mathbb{Q})\times (\mathbb{Z}/2\mathbb{Z})^r$$

→ If we can compute $J[2](\mathbb{Q})$, it is enough to find $|J(\mathbb{Q})/2J(\mathbb{Q})|$ to get r

2-descent: p-adic's help

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Idea: Consider p-adic numbers

 $C(\mathbb{F}_p)$ computable \rightsquigarrow points in $C(\mathbb{Q}_p)$ (Hensel) \rightsquigarrow points in $J(\mathbb{Q}_p)$ (Abel-Jacobi)

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Bonus: p-adic structure $|J(\mathbb{Q}_p)/2J(\mathbb{Q}_p)| = |J[2](\mathbb{Q}_p)|$ $(p \neq 2)$ $\rightsquigarrow J(\mathbb{Q}_p)/2J(\mathbb{Q}_p)$ computable

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Idea: Hasse's principle

Cohomology yields a finite group $\mathbf{Sel} = Sel^{(2)}(\mathbb{Q})$ s.t. $J(\mathbb{Q})/2J(\mathbb{Q}) \subset Sel$ with better computational properties.

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$$\begin{array}{ccccc} J(\mathbb{Q})/2J(\mathbb{Q}) & \hookrightarrow & H^{1}(\mathbb{Q}, J[2]) & \to & H^{1}(\mathbb{Q}, J) \\ \downarrow & & \downarrow res_{p} & & \downarrow \\ J(\mathbb{Q}_{p})/2J(\mathbb{Q}_{p}) & \hookrightarrow & H^{1}(\mathbb{Q}_{p}, J[2]) & \to & H^{1}(\mathbb{Q}_{p}, J) \end{array}$$

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definition: Selmer group

$$\begin{aligned} & \textit{Sel} := \{ \phi \in \textit{H}^1(\mathbb{Q},\textit{J[2]}) \mid \forall p, \textit{res}_p(\phi) \in \textit{J}(\mathbb{Q}_p)/2\textit{J}(\mathbb{Q}_p) \} \\ & := \bigcap_p \textit{res}_p^{-1}(\textit{J}(\mathbb{Q}_p)/2\textit{J}(\mathbb{Q}_p)) \end{aligned}$$

 $J(\mathbb{Q})/2J(\mathbb{Q}) \subset Sel \subset H^1(\mathbb{Q}, J[2])$ is a finite, but **abstract** group, equipped with a morphism, deduced from the **Weil Pairing**:

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- L*/(L*)² is effective:
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- $H^1(w)$ is injective (in some determined cases) in opposite with $J(\mathbb{Q})/2J(\mathbb{Q}) \to J(\mathbb{Q}_p)/2J(\mathbb{Q}_p)$

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$$\underbrace{\textit{Sel}}_{\substack{H^1(w)\\ }} wSel \subset \underbrace{\overbrace{L^*/(L^*)^2}^{effective}}_{\substack{effective\\ \\ L^*/(L^*)^2}}$$

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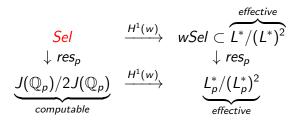
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Identification (from the def of Sel)

$$wSel = \{x \in L^*/(L^*)^2 \mid \forall p \ res_p(x) \in H^1(w)(J(\mathbb{Q}_p)/2J(\mathbb{Q}_p))\}$$

$$\cap ker(\mathcal{N})$$

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Solution (Hyperelliptic case)

If p is a **good reduction** prime($\neq 2$) and $H^1(w)$ is injective, $res_p(wSel) = ker(val_p) \cap ker(\mathcal{N})$

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 $S = \{ \text{primes of bad reduction} \} \cup \{2\}$ (finite) $\tilde{H} := (\bigcap_{p \notin S} ker(val_p)) \cap ker(\mathcal{N})$ finite dimension and computable

Selmer computation

 $wSel = \{x \in \tilde{H} \mid \forall p \in S \ res_p(x) \in H^1(w)(J(\mathbb{Q}_p)/2J(\mathbb{Q}_p))\}$ \rightsquigarrow finite amount of calculation

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(We obtain a finite amount of conditions because we know **exactly** the image of *wSel* by p-adic reduction with good primes)

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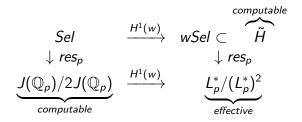
Conjecture

 $\operatorname{III}[2](\mathbb{Q})$ is "reasonably often" trivial

 \rightsquigarrow it is not very restrictive to only compute Sel \rightsquigarrow when it is not, try 3-descent

Reminder

 $S = \{ primes of bad reduction \}$



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• Find an explicit finite basis of $\tilde{H} \subset L^*/(L^*)^2$ (In practice: BNF on a field of degree $|J[2]| = 2^{2*g(C)}$)

• . . .

• Compute

 $wSel = \{x \in \tilde{H} \mid \forall p \in S \ res_p(x) \in H^1(w)(J(\mathbb{Q}_p)/2J(\mathbb{Q}_p))\}$

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• $dim_{\mathbb{F}_2}|J[2](\mathbb{Q})| + rank + dim_{\mathbb{F}_2}|\mathrm{III}[2](\mathbb{Q})|$ = $dim_{\mathbb{F}_2}|Sel| = dim_{\mathbb{F}_2}|wSel|$

Sum-up: Computable requirements for perfoming 2-descent

	Hyperelliptic	medium genus
$J[2](\overline{\mathbb{Q}}) + $ Galois action	\checkmark	Mascot
$J(\mathbb{Q}_p)/2J(\mathbb{Q}_p)$	\checkmark	\checkmark
$H^1(w)$ injective	or Stoll	?

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Problem 1

$$\begin{split} |H^{1}(J(\mathbb{Q}_{p})/2J(\mathbb{Q}_{p}))| &< |J[2](\mathbb{Q}_{p})| \\ &\rightsquigarrow \text{ we need to control} \\ KF_{p} := ker(J(\mathbb{Q}_{p})/2J(\mathbb{Q}_{p}) \xrightarrow{H^{1}(w)} L_{p}^{*}/(L_{p}^{*})^{2})) \end{split}$$

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Solution 1

 ${\it KF}_p$ could be controlled in practice if we can perform the division by 2

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Problem 2

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Solution 2: compute wSel

• Compute the old way an upper-bound:

 $wSel_{Fake} = \{ x \in \tilde{H} \mid \forall p \in S \ res_p(x) \in H^1(w)(J(\mathbb{Q}_p)/2J(\mathbb{Q}_p)) \}$

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- "effective Cebotarev theorem"
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- **Remark**: even if we reach *wSel*, we could probably not be able to detect it

3. $|wSel| \neq |Sel|$

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Problem 3

$$|J[2](\mathbb{Q})| + |J(\mathbb{Q})/2J(\mathbb{Q})| + |\mathrm{III}[2](\mathbb{Q})| = |Sel| = |wSel| \times |KF| \le |wSel_{Fake}| \times |KF|$$

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Solution 3

 $KF = (\bigcap_{p \in S} KF_p) \bigcap (\bigcap_{p \in A} KF_p)$ with: - $A \subset \{good \ primes\}$ is known if "Effective Cebotarev" - KF_p should be effectively computable if p is a good prime

Change of paradigm:

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We probably couldn't be able to certify |Sel| in the general case.

But we would try to be able to set several process aiming to narrow the bound of |Sel|, hoping for reaching a point low enough for our purposes (for instance lower than the genus in the Chabauty-Coleman frame)