# Introduction

In [1]:
```
5+7
```

Out[1]:
```
%1 = 12
```

In [2]:
```
1000!
```

Out[2]:
```
%2 = 4023872600770937735437024339230039857193748642107146325437999104299385123986290
205920442084869694048004799886101971960586316668729948085589013238296699445909974245
040870737599188236277271887325197795059509952761208749754624970436014182780946464962
910563938874378864873371191810458257836478499770124766328898359557354325131853239584
630755574091142624174743493475534286465766116677973966688202912073791438537195882498
081268678383745597317461360853795345242215865932019280908782973084313928444032812315
586110369768013573042161687476096758713483120254785893207671691324484262361314125087
802080002616831510273418279777047846358681701643650241536913982812648102130927612448
963599287051149649754199093422215668325720808213331861168115536158365469840467089756
029009505376164758477284218896796462449451607653534081989013854424879849599533191017
233555566021394503997362807501378376153071277619268490343526252000158885351473316117
021039681759215109077880193931781141945452572238655414610628921879602238389714760885
062768629671466746975629112340824392081601537808898939645182632436716167621791689097
799119037540312746222899880051954444142820121873617459926429565817466283029555702990
243241531816172104658320367869061172601587835207515162842255402651704833042261439742
869330616908979684825901254583271682264580665267699586526822728070757813918581788896
522081643483448259932660433676601769996128318607883861502794659551311565520360939881
806121385586003014356945272242063446317974605946825731037900840244324384656572450144
028218852524709351906209290231364932734975655139587205596542287497740114133469627154
228458623773875382304838656889764619273838149001407673104466402598994902222217659043
399018860185665264850617997023561938970178600408118897299183110211712298459016419210
688843871218556461249607987229085192968193723886426148396573822911231250241866493531
439701374285319266498753372189406942814341185201580141233448280150513996942901534830
776445690990731524332782882698646027898643211390835062170950025973898635542771967428
222487575867657523442202075736305694988250879689281627538488633969099598262809561214
509948717012445164612603790293091208890869420285106401821543994571568059418727489980
942547421735824010636774045957417851608292301353580818400969963725242305608559037006
242712434169090041536901059339838357779394109700277534720000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000
```

In [3]:
```
(1+x)^10
```

Out[3]:
```
%3 = x^10 + 10*x^9 + 45*x^8 + 120*x^7 + 210*x^6 + 252*x^5 + 210*x^4 + 120*x^3 + 45*x
^2 + 10*x + 1
```

In [4]:
```
exp(1)
```

Out[4]:
```
%4 = 2.7182818284590452353602874713526624978
```

In [5]:
```
factor(2^128+1)
```

Out[5]:
```
%5 =
[     59649589127497217 1]

[5704689200685129054721 1]
```

In [6]:
```
factor(x^4+4)
```

```
Out[6]:  %6 =
         [x^2 - 2*x + 2 1]

         [x^2 + 2*x + 2 1]
```

```
In [7]:  intnum(x=-oo,oo,1/(1+x^2))
```

```
Out[7]:  %7 = 3.1415926535897932384626433832795028842
```

```
In [8]:  sin(x)
```

```
Out[8]:  %8 = x - 1/6*x^3 + 1/120*x^5 - 1/5040*x^7 + 1/362880*x^9 - 1/39916800*x^11 + 1/62270
         20800*x^13 - 1/1307674368000*x^15 + O(x^17)
```

```
In [9]:  sin(x+O(x^20))
```

```
Out[9]:  %9 = x - 1/6*x^3 + 1/120*x^5 - 1/5040*x^7 + 1/362880*x^9 - 1/39916800*x^11 + 1/62270
         20800*x^13 - 1/1307674368000*x^15 + 1/355687428096000*x^17 - 1/121645100408832000*x^
         19 + O(x^20)
```

```
In [10]:  s=sqrt(2+O(7^10))
```

```
Out[10]:  %10 = 3 + 7 + 2*7^2 + 6*7^3 + 7^4 + 2*7^5 + 7^6 + 2*7^7 + 4*7^8 + 6*7^9 + O(7^10)
```

```
In [11]:  s^2
```

```
Out[11]:  %11 = 2 + O(7^10)
```

# Théorie de Galois élémentaire

```
In [12]:  P=x^4+1
```

```
Out[12]:  %12 = x^4 + 1
```

```
In [13]:  G=galoisinit(P);
```

```
In [14]:  [d,t]=galoisidentify(G)
```

```
Out[14]:  %14 = [4, 2]
```

```
In [15]:  galoisgetname(d,t)
```

```
Out[15]:  %15 = "C2 x C2"
```

```
In [16]:  S=galoissubgroups(G);
```

```
In [17]:  F=galoissubfields(G,1)
```

```
Out[17]:  %17 = [x, x^2 - 2, x^2 + 4, x^2 + 2, x^4 + 1]
```

```
In [18]:  galoisfixedfield(G,S[2],2)
```

```
Out[18]:  %18 = [y^2 - 2, Mod(-x^3 + x, x^4 + 1), [x^2 - y*x + 1, x^2 + y*x + 1]]
```

Nous avons la factorisation $x^4+1=(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$

```
In [19]:  Mod(-x^3 + x, x^4 + 1)^2
```

```
Out[19]:  %19 = Mod(2, x^4 + 1)
```

```
In [20]: galoisfixedfield(G,S[3],2)
```

```
Out[20]: %20 = [y^2 + 4, Mod(2*x^2, x^4 + 1), [x^2 - 1/2*y, x^2 + 1/2*y]]
```

Nous avons la factorisation $x^4+1=(x^2-\sqrt{-1})(x^2+\sqrt{-1})$

```
In [21]: galoisfixedfield(G,S[4],2)
```

```
Out[21]: %21 = [y^2 + 2, Mod(x^3 + x, x^4 + 1), [x^2 - y*x - 1, x^2 + y*x - 1]]
```

Nous avons la factorisation $x^4+1=(x^2-\sqrt{-2}x-1)(x^2+\sqrt{-2}x-1)$

# Théorie algébrique des nombres

```
In [22]: K=bnfinit(alpha^2-79);
```

Initialise $K=\mathbb{Q}(\alpha)$ avec $\alpha=\sqrt{79}$

```
In [23]: K.zk
```

```
Out[23]: %23 = [1, alpha]
```

$\mathcal{O}_K=\mathbb{Z}+\alpha\mathbb{Z}=\mathbb{Z}[\alpha]$

```
In [24]: K.disc
```

```
Out[24]: %24 = 316
```

Le discriminant de $K$ vaut $316$

```
In [25]: K.sign
```

```
Out[25]: %25 = [2, 0]
```

$K$ est totalement réel.

```
In [26]: K.cyc
```

```
Out[26]: %26 = [3]
```

Le groupe de classe est $\mathbb{Z}/3\mathbb{Z}$.

```
In [27]: K.fu
```

```
Out[27]: %27 = [Mod(9*alpha + 80, alpha^2 - 79)]
```

L'unité fondamentale est $80-9\alpha$.

```
In [28]: 80^2-79*9^2
```

```
Out[28]: %28 = 1
```

```
In [29]: K.reg
```

```
Out[29]: %29 = 5.0751347504448098597876951184786988027
```

Le régulateur vaut $\simeq 5.075134750444809$

```
In [30]: log(K.fu[1])
```

```
Out[30]: %30 = [-5.0751347504448098597876951184786988598, 5.075134750444809859787695118478698
         8038]~
```

```
In [31]: id2 = idealprimedec(K,2);
```

id2 est la décomposition de $2$ dans le corps $K$.

```
In [32]: g2=#id2
```

```
Out[32]: %32 = 1
```

```
In [33]: id2[1].e
```

```
Out[33]: %33 = 2
```

```
In [34]: id2[1].f
```

```
Out[34]: %34 = 1
```

$2\mathcal{O}_K = \mathfrak{p}_2^2$

```
In [35]: bnfisprincipal(K,id2[1])
```

```
Out[35]: %35 = [[0]~, [-9, -1]~]
```

$\mathfrak{p}_2 = (9+\alpha)$

```
In [36]: 9^2-79*1^2
```

```
Out[36]: %36 = 2
```

```
In [37]: id3 = idealprimedec(K,3);
```

```
In [38]: g3=#id3
```

```
Out[38]: %38 = 2
```

```
In [39]: id3[1].e
```

```
Out[39]: %39 = 1
```

```
In [40]: id3[1].f
```

```
Out[40]: %40 = 1
```

$3\mathcal{O}_K = \mathfrak{p}_3\mathfrak{p}_3'$

```
In [41]:  bnfisprincipal(K,id3[1])
```

```
Out[41]: %41 = [[1]~, [1, 0]~]
```

$\mathfrak{p}_3$ n'est pas principal

```
In [42]: p3 = idealpow(K, id3[1], 3);
```

p3=$\mathfrak{p}_3^3$

```
In [43]: bnfisprincipal(K,p3)
```

```
Out[43]:  %43 = [[0]~, [-17, 2]~]
```

$\mathfrak{p}_3^3$ est principal et engendré par $2\alpha - 17$.

```
In [44]:  17^2-79*2^2

Out[44]:  %44 = -27

In [45]:  id5 = idealprimedec(K,5);

In [46]:  g5 = #id5

Out[46]:  %46 = 2

In [47]:  id5[1].e

Out[47]:  %47 = 1

In [48]:  id5[1].f

Out[48]:  %48 = 1
```

$$5\mathcal{O}_K = \mathfrak{p}_5\mathfrak{p}_5'$$

```
In [49]:  bnfisprincipal(K,id5[1])

Out[49]:  %49 = [[1]~, [-8/3, 1/3]~]
```

$\mathfrak{p}_5$ est dans la même classe d'idéal que $\mathfrak{p}_3$.

```
In [50]:  p35 = idealmul(K, id3[1], id5[2]);
```

p35=$\mathfrak{p}_3\mathfrak{p}_5'$

```
In [51]:  bnfisprincipal(K,p35)

Out[51]:  %51 = [[0]~, [8, 1]~]
```

$$\mathfrak{p}_3\mathfrak{p}_5' = (-8 - \alpha)$$

```
In [52]:  8^2-79*1^2

Out[52]:  %52 = -15

In [53]:  bnfisintnorm(K,1049)

Out[53]:  %53 = [13*alpha + 120, 40*alpha + 357]

In [54]:  120^2-79*13^2

Out[54]:  %54 = 1049
```

# Fonctions L

```
In [55]:  Z=lfuninit(K,[1/2,1/2,100],1);
```

Z=$\zeta_K(s)=\prod_{\mathfrak{p}}\frac{1}{1-\mathrm{Norm}(\mathfrak{p})^{-s}}$ la fonction zêta de Dedekind de $K$

```
In [56]: lfun(Z,0,0)
```

```
Out[56]: %56 = 0
```

```
In [57]: lfun(Z,0,1)
```

```
Out[57]: %57 = -7.6127021256672147896815426777180482058
```

```
In [58]: -K.no*K.reg/2
```

```
Out[58]: %58 = -7.6127021256672147896815426777180482040
```

L'égalité vient de la formule du nombre de classe de Dirichlet.

```
In [59]: lfun(Z,1)
```

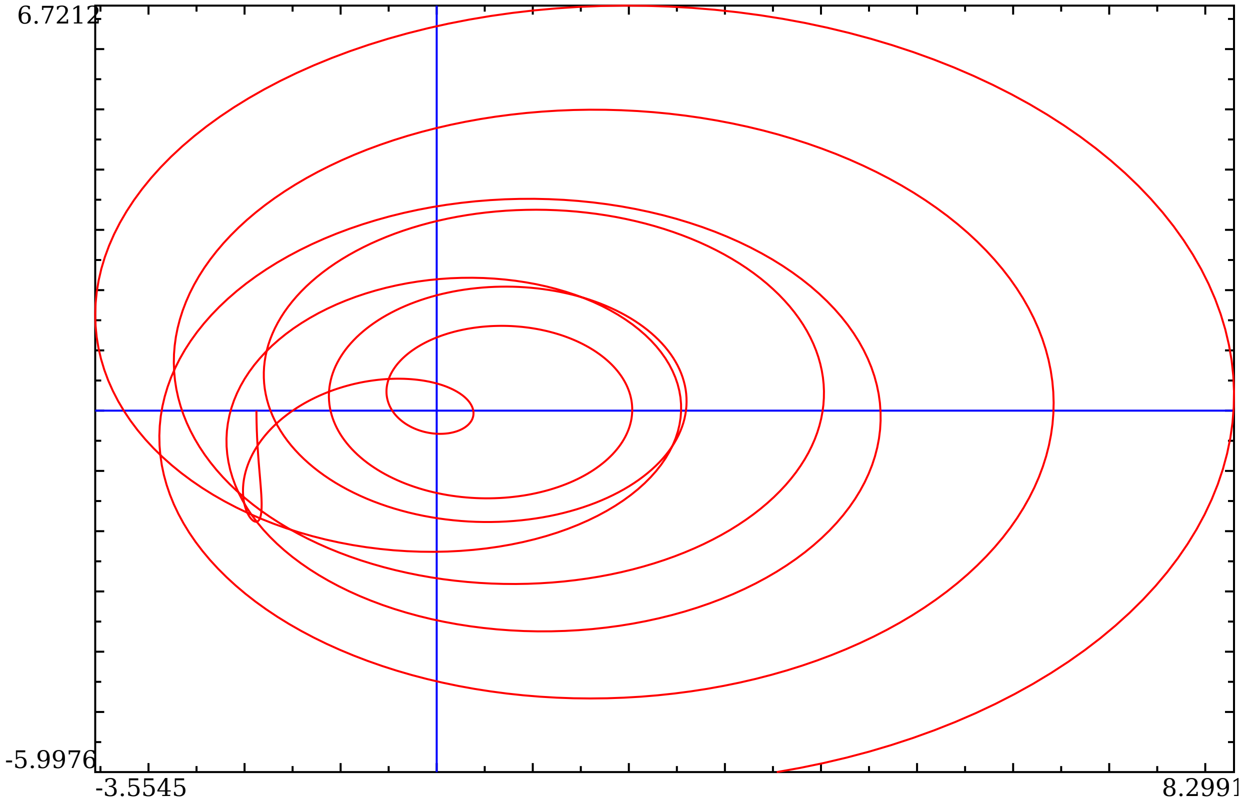
```
Out[59]: %59 = 1.7129918109884013874307672625564043042*x^-1 + O(x^0)
```

```
In [60]: 2^K.r1*K.no*K.reg/sqrt(K.disc)/2
```

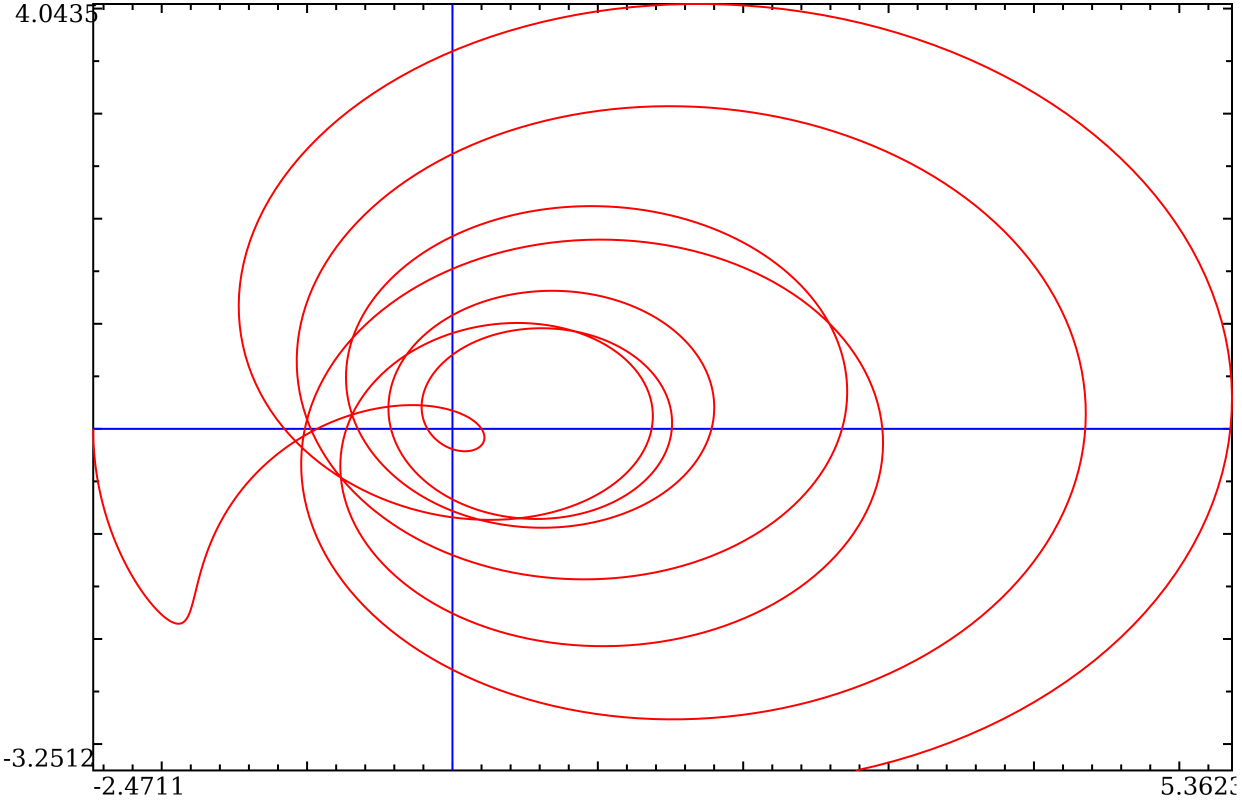
```
Out[60]: %60 = 1.7129918109884013874307672625564043038
```

L'égalité vient aussi de la formule du nombre de classe de Dirichlet.

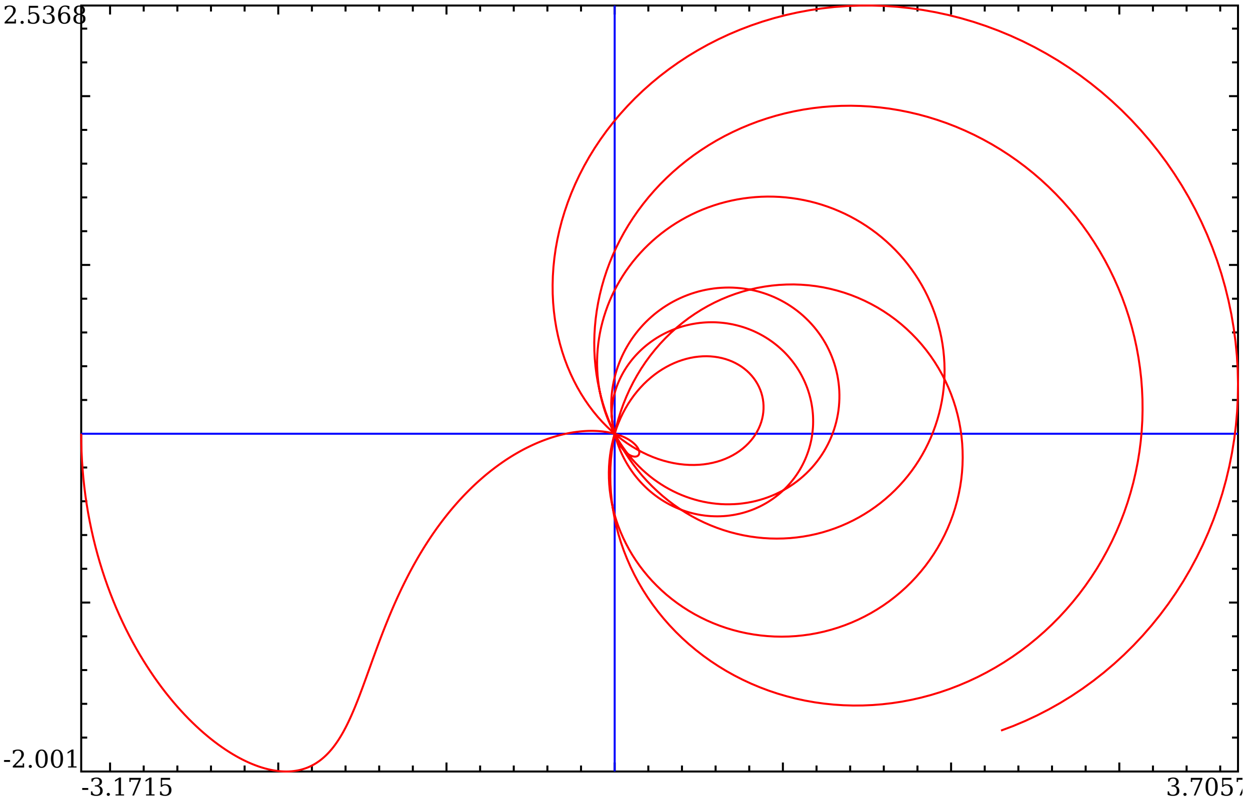
```
In [61]: ploth(x=0,10,lfun(Z,.3+I*x),"Complex");
```



```
In [62]: ploth(x=0,10,lfun(Z,.4+I*x),"Complex");
```



```
In [63]: ploth(x=0,10,lfun(Z,.5+I*x),"Complex");
```



L'hypothèse de Riemann pour $\zeta_K$ est que les zéros sont de partie réel 1/2.

# Théorie du corps de classes

```
In [64]: B=bnrclassfield(K)
```

```
Out[64]: %64 = [x^3 - 21*x - 4*alpha]
```

```
In [65]: L=rnfinit(K,B[1]);
```

```
In [66]: L.disc
```

```
Out[66]: %66 = [1, 1]
```

L'idéal discriminant relatif vaut 1.

```
In [67]: rnfisabelian(K,B[1])
```

```
Out[67]: %67 = 1
```

L'extension est abélienne.

```
In [68]: p1049=idealprimedec(K,1049);
```

```
In [69]: #p1049
```

```
Out[69]: %69 = 2
```

1049 est totalement décomposé dans K

```
In [70]: #rnfidealprimedec(L,p1049[1])
```

```
Out[70]: %70 = 3
```

$\mathfrak{p}_{1049}$ est totalement décomposé dans $L$ et donc principal.

```
In [71]: bnfisprincipal(K,p1049[1])
```

```
Out[71]: %71 = [[0]~, [120, -13]~]
```

# Équations diophantiennes

Résoudre $x^2 - 6712y^2 = 1$ avec $x$, $y$ entiers.

```
In [72]: bnf=bnfinit(alpha^2-6712);bnf.fu
```

```
Out[72]: %72 = [Mod(1193432464696042980*alpha + 97774098782702401430, alpha^2 - 6712)]
```

```
In [73]: 97774098782702401430^2-6712*1193432464696042980^2
```

```
Out[73]: %73 = 1
```

Résoudre $y^2 = x^3 - 157^2 x$, avec $x$, $y$ rationels.

```
In [74]: E=ellinit([-157^2,0]); P=ellheegner(E)
```

```
Out[74]: %74 = [69648970982596494254458225/166136231668185267540804, 538962435089604615078004
          307258785218335/67716816556077455999228495435742408]
```

```
In [75]: P[1]^3-157^2*P[1] - P[2]^2
```

```
Out[75]: %75 = 0
```

Résoudre $x^5 - 2y^5 = 3$ avec $x$, $y$ entiers.

```
In [76]: thue(x^5-2,3)
```

```
Out[76]: %76 = [[1, -1]]
```

Il y a seulement la solution triviale $x = 1$, $y = -1$

```
In [77]: bnf=bnfinit(alpha^3-2);
```

Résoudre $x^3 + 2y^3 + 4z^3 - 6xyz = 10009$

```
In [78]: bnfisintnorm(bnf,10009)

Out[78]: %78 = [-6*alpha^2 + 13*alpha + 11, 13*alpha^2 + 10*alpha + 1, -6*alpha^2 + 15*alpha
          + 7]

In [79]: 11^3+2*13^3-4*6^3-6*11*13*-6

Out[79]: %79 = 10009
```

# Interpolation

```
In [80]: exp(Pi*sqrt(163))

Out[80]: %80 = 262537412640768743.99999999999925007259

 In [ ]:

In [81]: 744-ellj((1+sqrt(-163))/2)

Out[81]: %81 = 262537412640768744.00000000000000000000

In [82]: ellj((1+sqrt(-31))/2)

Out[82]: %82 = -39492793.911556244143880327445303424864

In [83]: algdep(%,3)

Out[83]: %83 = 214732*x^3 + 8480366314537*x^2 - 12151115349442*x - 13611811239453

In [84]: localprec(100); z=ellj((1+sqrt(-31))/2)

Out[84]: %84 = -39492793.911556244143880327445303424866

In [85]: algdep(z,3)

Out[85]: %85 = x^3 + 39491307*x^2 - 58682638134*x + 1566028350940383

In [86]: polclass(-31)

Out[86]: %86 = x^3 + 39491307*x^2 - 58682638134*x + 1566028350940383

In [87]: z=sumpos(n=1,sumpos(m=n+1,1/(n^2*m^5)))

Out[87]: %87 = 0.038575124342753255505925464372562108149

In [88]: lindep([z,zeta(7),zeta(5)*zeta(2),zeta(3)*zeta(4)])

Out[88]: %88 = [1, 11, -5, -2]~

In [89]: 5*zeta(5)*zeta(2)+2*zeta(3)*zeta(4)-11*zeta(7)

Out[89]: %89 = 0.038575124342753255505925464372995570013

In [90]: 1/4+O(5^20)
```

```
Out[90]:  %90 = 4 + 3*5 + 3*5^2 + 3*5^3 + 3*5^4 + 3*5^5 + 3*5^6 + 3*5^7 + 3*5^8 + 3*5^9 + 3*5^
          10 + 3*5^11 + 3*5^12 + 3*5^13 + 3*5^14 + 3*5^15 + 3*5^16 + 3*5^17 + 3*5^18 + 3*5^19
           + O(5^20)
```

```
In [91]:  s=gamma(1/4+O(5^20))
```

```
Out[91]:  %91 = 1 + 4*5 + 3*5^4 + 5^6 + 5^7 + 4*5^9 + 5^10 + 2*5^12 + 5^13 + 2*5^14 + 5^15 + 3
          *5^16 + 2*5^18 + 3*5^19 + O(5^20)
```

```
In [92]:  algdep(s,4)
```

```
Out[92]:  %92 = x^4 + 4*x^2 + 5
```

```
In [93]:  s^4+4*s^2+5
```

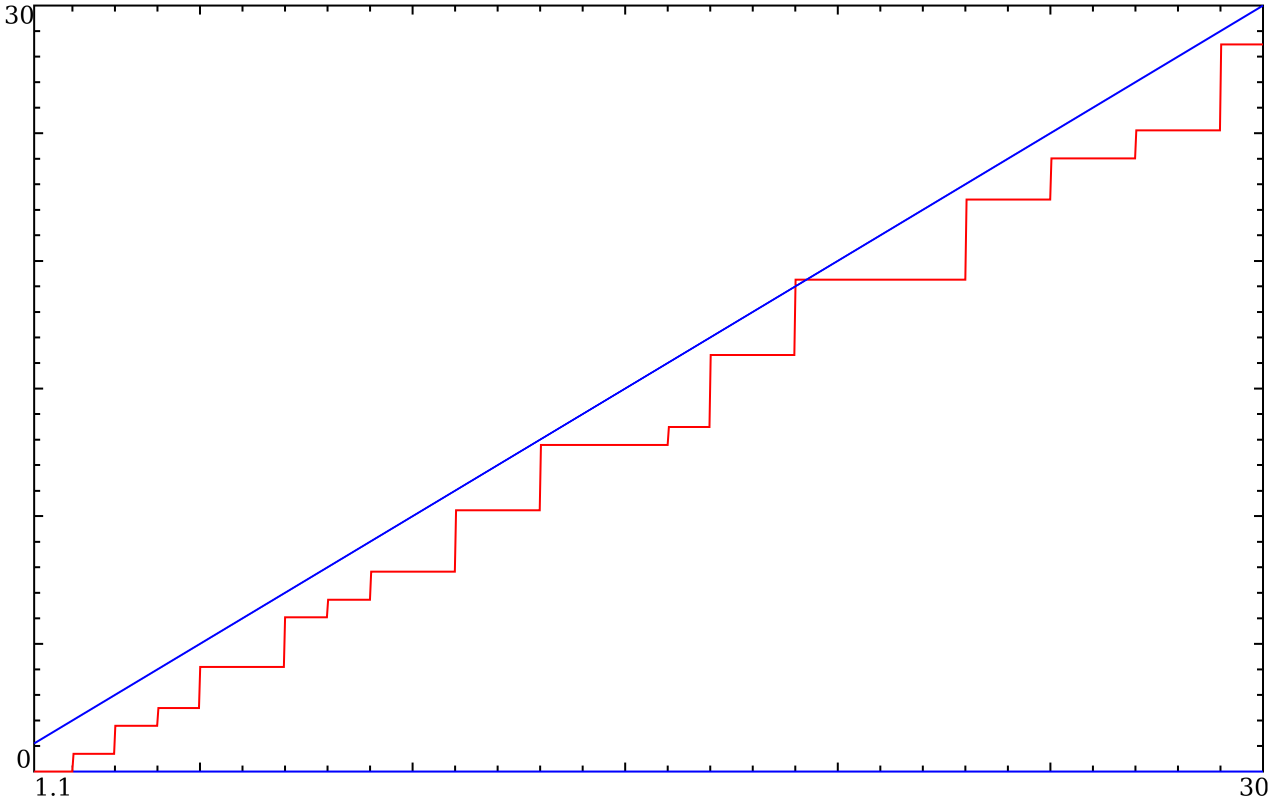
```
Out[93]:  %93 = O(5^20)
```

# Formule explicite de Riemann

```
In [94]:  default(graphcolors,[4,2,5]);
```

```
In [95]:  Cpsi(n)=log(lcm([1..floor(n)]));
```

```
In [96]:  ploth(x=1.1,30,[Cpsi(x),x]);
```

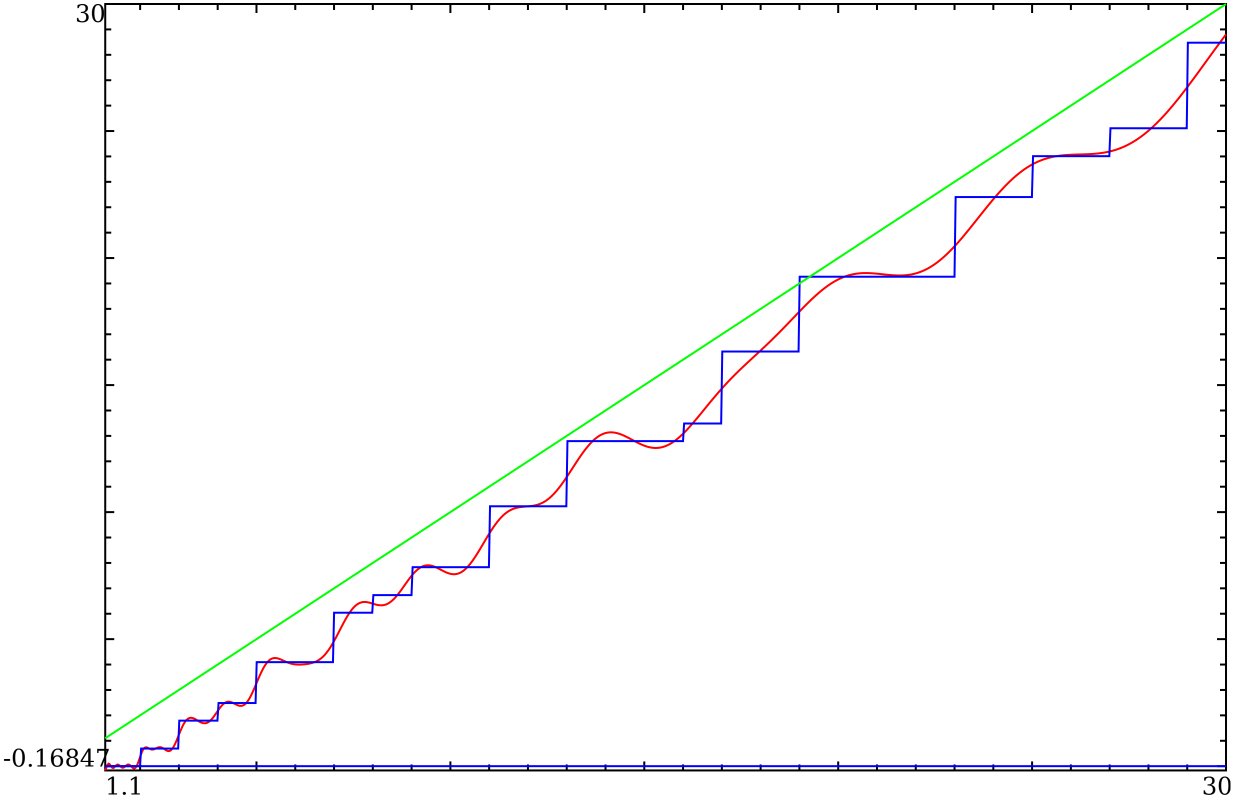


```
In [97]:  Z=lfunzeros(1,1000);
```

```
In [98]:  F(x,s=#Z)=my(lx=I*log(x));x-2*sqrt(x)*real(sum(i=1,s,exp(lx*Z[i])/(1/2+I*Z[i])))-log(
```

Fonction $\psi$ de Chebyshev

```
In [110…  ploth(x=1.1,30,[F(x,5),Cpsi(x),x])
```



Out[110…

```
%106 = [1.1000000000000000888, 30.000000000000000000, -0.16847039935043686687, 30.00
0000000000000000]
```

In [ ]: