IntroTNA

January 11, 2024

1 Introduction

[1]:
```
5+7
```

[1]:
```
%1 = 12
```

[2]:
```
1000!
```

[2]:
```
%2 = 4023872600770937735437024339230039857193748642107146325437999104299385123986
2902059204420848696940480047998861019719605863166687299480855890132382966994459
0997424504087073759918823627727188732519779505950995276120874975462497043601418
2780946464962910563938874378864873371191810458257836478499770124766328898359557
3543251318532395846307555740911426241747434934755342864657661166779739666882029
1207379143853719588249808126867838374559731746136085379534524221586593201928090
8782973084313928444032812315586110369768013573042161687476096758713483120254785
8932076716913244842623613141250878020800026168315102734182797770478463586817016
4365024153691398281264810213092761244896359928705114964975419909342221566832572
0808213331861168115536158365469840467089756029009505376164758477284218896796462
4494516076535340819890138544248798495995331910172335555660213945039973628075013
7837615307127761926849034352625200015888535147331611702103968175921510907788019
3931781141945452572238655414610628921879602238389714760885062768629671466746975
6291123408243920816015378088989396451826324367161676217916890977991190375403127
4622289988005195444414282012187361745992642956581746628302955570299024324153181
6172104658320367869061172601587835207515162842255402651704833042261439742869330
6169089796848259012545832716822645806652676995865268227280707578139185817888965
2208164348344825993266043367660176999612831860788386150279465955131156552036093
9881806121385586003014356945272242063446317974605946825731037900840244324384656
5724501440282188525247093519062092902313649327349756551395872055965422874977401
1413346962715422845862377387538230483865688976461927383814900140767310446640259
8994902222217659043399018860185665264850617997023561938970178600408118897299183
1102117122984590164192106888438712185564612496079872290851929681937238864261483
9657382291123125024186649353143970137428531926649875337218940694281434118520158
0141233448280150513996942901534830776445690990731524332782882698646027898643211
3908350621709500259738986355427719674282224875758676575234422020757363056949882
5087968928162753848863396909959826280956121450994871701244516461260379029309120
8890869420285106401821543994571568059418727489980942547421735824010636774045957
4178516082923013535808184009699637252423056085590370062427124341690900415369010
5933983835777939410970027753
```

```
34720000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000
0000000000000
```

[3]:
```
(1+x)^10
```

[3]:
```
%3 = x^10 + 10*x^9 + 45*x^8 + 120*x^7 + 210*x^6 + 252*x^5 + 210*x^4 + 120*x^3 +
45*x^2 + 10*x + 1
```

[4]:
```
exp(1)
```

[4]:
```
%4 = 2.7182818284590452353602874713526624978
```

[5]:
```
factor(2^128+1)
```

[5]:
```
%5 =
[      59649589127497217 1]

[5704689200685129054721 1]
```

[6]:
```
factor(x^4+4)
```

[6]:
```
%6 =
[x^2 - 2*x + 2 1]

[x^2 + 2*x + 2 1]
```

[7]:
```
intnum(x=-oo,oo,1/(1+x^2))
```

[7]:
```
%7 = 3.1415926535897932384626433832795028842
```

[8]:
```
sin(x)
```

[8]:
```
%8 = x - 1/6*x^3 + 1/120*x^5 - 1/5040*x^7 + 1/362880*x^9 - 1/39916800*x^11 +
1/6227020800*x^13 - 1/1307674368000*x^15 + O(x^17)
```

[9]:
```
sin(x+O(x^20))
```

[9]: %9 = x - 1/6*x^3 + 1/120*x^5 - 1/5040*x^7 + 1/362880*x^9 - 1/39916800*x^11 +
1/6227020800*x^13 - 1/1307674368000*x^15 + 1/355687428096000*x^17 -
1/121645100408832000*x^19 + O(x^20)

[10]:
```
s=sqrt(2+O(7^10))
```

[10]: %10 = 3 + 7 + 2*7^2 + 6*7^3 + 7^4 + 2*7^5 + 7^6 + 2*7^7 + 4*7^8 + 6*7^9 +
O(7^10)

[11]:
```
s^2
```

[11]: %11 = 2 + O(7^10)

2 Théorie de Galois élémentaire

[12]:
```
P=x^4+1
```

[12]: %12 = x^4 + 1

[13]:
```
G=galoisinit(P);
```

[14]:
```
[d,t]=galoisidentify(G)
```

[14]: %14 = [4, 2]

[15]:
```
galoisgetname(d,t)
```

[15]: %15 = "C2 x C2"

[16]:
```
S=galoissubgroups(G);
```

[17]:
```
F=galoissubfields(G,1)
```

[17]: %17 = [x, x^2 - 2, x^2 + 4, x^2 + 2, x^4 + 1]

[18]:
```
galoisfixedfield(G,S[2],2)
```

[18]: %18 = [y^2 - 2, Mod(-x^3 + x, x^4 + 1), [x^2 - y*x + 1, x^2 + y*x + 1]]

Nous avons la factorisation $x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$

[19]:
```
Mod(-x^3 + x, x^4 + 1)^2
```

[19]: %19 = Mod(2, x^4 + 1)

[20]:
```
galoisfixedfield(G,S[3],2)
```

[20]: %20 = [y^2 + 4, Mod(2*x^2, x^4 + 1), [x^2 - 1/2*y, x^2 + 1/2*y]]

Nous avons la factorisation $x^4 + 1 = (x^2 - \sqrt{-1})(x^2 + \sqrt{-1})$

[21]:
```
galoisfixedfield(G,S[4],2)
```

[21]: %21 = [y^2 + 2, Mod(x^3 + x, x^4 + 1), [x^2 - y*x - 1, x^2 + y*x - 1]]

Nous avons la factorisation $x^4 + 1 = (x^2 - \sqrt{-2}x - 1)(x^2 + \sqrt{-2}x - 1)$

3 Théorie algébrique des nombres

[22]:
```
K=bnfinit(alpha^2-79);
```

Initialise $K = \mathbb{Q}(\alpha)$ avec $\alpha = \sqrt{79}$

[23]:
```
K.zk
```

[23]: %23 = [1, alpha]

$\mathcal{O}_K = \mathbb{Z} + \alpha\mathbb{Z} = \mathbb{Z}[\alpha]$

[24]:
```
K.disc
```

[24]: %24 = 316

Le discriminant de K vaut 316

[25]:
```
K.sign
```

[25]: %25 = [2, 0]

K est totalement réel.

[26]:
```
K.cyc
```

[26]: %26 = [3]

Le groupe de classe est $\mathbb{Z}/3\mathbb{Z}$.

[27]:
```
K.fu
```

[27]:
```
%27 = [Mod(-9*alpha + 80, alpha^2 - 79)]
```

L'unité fondamentale est $80 - 9\alpha$.

[28]:
```
80^2-79*9^2
```

[28]:
```
%28 = 1
```

[29]:
```
K.reg
```

[29]:
```
%29 = 5.0751347504448098597876951184786988038
```

Le régulateur vaut $\simeq 5.075134750444809$

[30]:
```
log(K.fu[1])
```

[30]:
```
%30 = [5.0751347504448098597876951184786988038,
-5.0751347504448098597876951184786988598]~
```

[31]:
```
id2 = idealprimedec(K,2);
```

id2 est la décomposition de 2 dans le corps K.

[32]:
```
g2=#id2
```

[32]:
```
%32 = 1
```

[33]:
```
id2[1].e
```

[33]:
```
%33 = 2
```

[34]:
```
id2[1].f
```

[34]:
```
%34 = 1
```

$2\mathcal{O}_K = \mathfrak{p}_2^2$

[35]:
```
bnfisprincipal(K,id2[1])
```

[35]: %35 = [[0]~, [9, 1]~]

$\mathfrak{p}_2 = (9+\alpha)$

[36]:
```
9^2-79*1^2
```

[36]: %36 = 2

[37]:
```
id3 = idealprimedec(K,3);
```

[38]:
```
g3=#id3
```

[38]: %38 = 2

[39]:
```
id3[1].e
```

[39]: %39 = 1

[40]:
```
id3[1].f
```

[40]: %40 = 1

$3\mathcal{O}_K = \mathfrak{p}_3\mathfrak{p}_3'$

[41]:
```
 bnfisprincipal(K,id3[1])
```

[41]: %41 = [[1]~, [1, 0]~]

$\mathfrak{p}_3$ n'est pas principal

[42]:
```
p3 = idealpow(K, id3[1], 3);
```

p3=$\mathfrak{p}_3^3$

[43]:
```
bnfisprincipal(K,p3)
```

[43]: %43 = [[0]~, [-17, 2]~]

$\mathfrak{p}_3^3$ est principal et engendré par $2\alpha - 17$.

[44]:
```
17^2-79*2^2
```

[44]: %44 = -27

[45]:
```
id5 = idealprimedec(K,5);
```

[46]:
```
g5 = #id5
```

[46]: %46 = 2

[47]:
```
id5[1].e
```

[47]: %47 = 1

[48]:
```
id5[1].f
```

[48]: %48 = 1

$5\mathcal{O}_K = \mathfrak{p}_5\mathfrak{p}_5'$

[49]:
```
bnfisprincipal(K,id5[1])
```

[49]: %49 = [[1]~, [-8/3, 1/3]~]

$\mathfrak{p}_5$ est dans la même classe d'idéal que $\mathfrak{p}_3$.

[50]:
```
p35 = idealmul(K, id3[1], id5[2]);
```

p35=$\mathfrak{p}_3\mathfrak{p}_5'$

[51]:
```
bnfisprincipal(K,p35)
```

[51]: %51 = [[0]~, [-8, -1]~]

$\mathfrak{p}_3\mathfrak{p}_5' = (-8-\alpha)$

[52]:
```
8^2-79*1^2
```

[52]: %52 = -15

[53]:
```
bnfisintnorm(K,1049)
```

[53]: %53 = [13*alpha + 120, 40*alpha + 357]

[54]:
```
120^2-79*13^2
```

[54]: %54 = 1049

4 Fonctions L

[55]:
```
Z=lfuninit(K,[1/2,1/2,100],1);
```

Z=$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1-\mathrm{Norm}(\mathfrak{p})^{-s}}$ la fonction zêta de Dedekind de K

[56]:
```
lfun(Z,0,0)
```

[56]: %56 = 0

[57]:
```
lfun(Z,0,1)
```

[57]: %57 = -7.6127021256672147896815426777180482058

[58]:
```
-K.no*K.reg/2
```

[58]: %58 = -7.6127021256672147896815426777180482058

L'égalité vient de la formule du nombre de classe de Dirichlet.

[59]:
```
lfun(Z,1)
```

[59]: %59 = 1.7129918109884013874307672625564043042*x^-1 + O(x^0)

[60]:
```
2^K.r1*K.no*K.reg/sqrt(K.disc)/2
```

[60]: %60 = 1.7129918109884013874307672625564043042

L'égalité vient aussi de la formule du nombre de classe de Dirichlet.

[61]:
```
ploth(x=0,10,lfun(Z,.3+I*x),"Complex");
```

[62]:
```
ploth(x=0,10,lfun(Z,.4+I*x),"Complex");
```

[63]:
```
ploth(x=0,10,lfun(Z,.5+I*x),"Complex");
```

L'hypothèse de Riemann pour ζ_K est que les zéros sont de partie réel 1/2.

5 Théorie du corps de classes

[64]:
```
B=bnrclassfield(K)
```

[64]:
```
%64 = [x^3 + (2274*alpha - 20220)*x + (105464*alpha - 937376)]
```

[65]:
```
L=rnfinit(K,B[1]);
```

[66]:
```
L.disc
```

[66]:
```
%66 = [1, [4095520, -460782]~]
```

L'idéal discriminant relatif vaut 1.

[67]:
```
rnfisabelian(K,B[1])
```

[67]:
```
%67 = 1
```

L'extension est abélienne.

[68]:
```
p1049=idealprimedec(K,1049);
```

[69]:
```
#p1049
```

[69]:
```
%69 = 2
```

1049 est totalement décomposé dans K

[70]:
```
#rnfidealprimedec(L,p1049[1])
```

[70]:
```
%70 = 3
```

$\mathfrak{p}_{1049}$ est totalement décomposé dans L et donc principal.

[71]:
```
bnfisprincipal(K,p1049[1])
```

[71]:
```
%71 = [[0]~, [120, -13]~]
```

6 Équations diophantiennes

Résoudre $x^2 - 6712y^2 = 1$ avec x, y entiers.

[72]:
```
bnf=bnfinit(alpha^2-6712);bnf.fu
```

[72]: %72 = [Mod(-1193432464696042 98*alpha - 9777409878270240143, alpha^2 - 6712)]

[73]:
```
9777409878270240143^2-6712*119343246469604298^2
```

[73]: %73 = 1

Résoudre $y^2 = x^3 - 157^2 x$, avec x, y rationels.

[74]:
```
E=ellinit([-157^2,0]); P=ellheegner(E)
```

[74]: %74 = [69648970982596494254458225/166136231668185267540804,
538962435089604615078004307258785218335/67716816556077455999228495435742408]

[75]:
```
P[1]^3-157^2*P[1] - P[2]^2
```

[75]: %75 = 0

Résoudre $x^5 - 2y^5 = 3$ avec x, y entiers.

[76]:
```
thue(x^5-2,3)
```

[76]: %76 = [[1, -1]]

Il y a seulement la solution triviale $x = 1$, $y = -1$

[77]:
```
bnf=bnfinit(alpha^3-2);
```

Résoudre $x^3 + 2y^3 + 4z^3 - 6xyz = 10009$

[78]:
```
bnfisintnorm(bnf,10009)
```

[78]: %78 = [-6*alpha^2 + 13*alpha + 11, 13*alpha^2 + 10*alpha + 1, -6*alpha^2 +
15*alpha + 7]

[79]:
```
11^3+2*13^3-4*6^3-6*11*13*-6
```

[79]: %79 = 10009

7 Interpolation

[80]:
```
exp(Pi*sqrt(163))
```

[80]: %80 = 262537412640768743.99999999999925007259

```
[ ]:
```

```
[81]: 744-ellj((1+sqrt(-163))/2)
```

[81]: %81 = 262537412640768744.00000000000000000000

```
[82]: ellj((1+sqrt(-31))/2)
```

[82]: %82 = -39492793.911556244143880327445303424864

```
[83]: algdep(%,3)
```

[83]: %83 = 214732*x^3 + 8480366314537*x^2 - 12151115349442*x - 13611811239453

```
[84]: localprec(100); z=ellj((1+sqrt(-31))/2)
```

[84]: %84 = -39492793.911556244143880327445303424866

```
[85]: algdep(z,3)
```

[85]: %85 = x^3 + 39491307*x^2 - 58682638134*x + 1566028350940383

```
[86]: polclass(-31)
```

[86]: %86 = x^3 + 39491307*x^2 - 58682638134*x + 1566028350940383

```
[87]: z=sumpos(n=1,sumpos(m=n+1,1/(n^2*m^5)))
```

[87]: %87 = 0.038575124342753255505925464372562108149

```
[88]: lindep([z,zeta(7),zeta(5)*zeta(2),zeta(3)*zeta(4)])
```

[88]: %88 = [-1, -11, 5, 2]~

```
[89]: 5*zeta(5)*zeta(2)+2*zeta(3)*zeta(4)-11*zeta(7)
```

[89]: %89 = 0.038575124342753255505925464372995570013

[90]:
```
1/4+O(5^20)
```

[90]:
```
%90 = 4 + 3*5 + 3*5^2 + 3*5^3 + 3*5^4 + 3*5^5 + 3*5^6 + 3*5^7 + 3*5^8 + 3*5^9 +
3*5^10 + 3*5^11 + 3*5^12 + 3*5^13 + 3*5^14 + 3*5^15 + 3*5^16 + 3*5^17 + 3*5^18 +
3*5^19 + O(5^20)
```

[91]:
```
s=gamma(1/4+O(5^20))
```

[91]:
```
%91 = 1 + 4*5 + 3*5^4 + 5^6 + 5^7 + 4*5^9 + 5^10 + 2*5^12 + 5^13 + 2*5^14 + 5^15
+ 3*5^16 + 2*5^18 + 3*5^19 + O(5^20)
```

[92]:
```
algdep(s,4)
```

[92]:
```
%92 = x^4 + 4*x^2 + 5
```

[93]:
```
s^4+4*s^2+5
```

[93]:
```
%93 = O(5^20)
```

8 Formule explicite de Riemann

[94]:
```
Z=[1/2+I*t|t<-lfunzeros(1,1000)];
```

[95]:
```
F(x,s=#Z)=x-2*real(sum(i=1,s,x^Z[i]/Z[i]))-log(2*Pi)-1/2*log(1-x^-2);
```

[96]:
```
ploth(x=2,20,F(x,100));
```

Fonction ψ de Chebyshev

[97]:
```
Cpsi(n)=log(lcm([1..floor(n)]));
```

[98]:
```
default(graphcolors,[4,2,5]);
```

[99]:
```
ploth(x=2,20,[F(x,30),Cpsi(x),x]);
```