[Tutorial] The Modular Forms Package

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PARI GP

Basic Implementation

We will work with modular forms in spaces $M_k(\Gamma_0(N), \chi)$, where χ is a Dirichlet character modulo N and k is integral or half-integral. Three types of objects:

- Modular form spaces, initialized by the command `mfinit` with a flag specifying which subspace of M_k we want to work in (S_k^{new}, S_k, S_k^{old}, $\mathcal{E}_k$, M_k).
- Modular forms themselves: if `F` is such a form, `mfcoefs(F,n)` gives the vector of coefficients $[a(0), a(1), ..., a(n)]$, and `mfparams(F)` gives $[N, k, \chi, pol]$, level, weight, character, and polynomial in y defining the field $\mathbb{Q}(F)/\mathbb{Q}(\chi)$.
- Dirichlet characters: represented either by a discriminant D for the Kronecker–Legendre symbol (D/n) ($D = 1$ trivial character), by an `intmod` `Mod(a,N)` with $\gcd(a, N) = 1$ (Conrey numbering), or by a general `Pari/GP` group $[G, \chi]$.

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Modular Form Leaves I

```
D = mfDelta(); V = mfcoefs(D, 8)
Ser(V,q)
```

```
% = [0, 1, -24, 252, -1472, 4830, -6048, -16744, 84480]
% = q - 24*q^2 + 252*q^3 - 1472*q^4 + 4830*q^5 - 6048*q^6
    - 16744*q^7 + 84480*q^8 + O(q^9)
```

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      - 16744*q^7 + 84480*q^8 + O(q^9)
```

Modular Form Leaves II

```
E4 = mfEk(4); E6 = mfEk(6);
apply(x->mfcoefs(x,4),[E4,E6])
E43 = mfpow(E4, 3); E62 = mfpow(E6, 2);
DP = mflinear([E43, E62], [1, -1]/1728);
mfcoefs(DP, 6)
mfisequal(D, DP)
```

```
% = [[1, 240, 2160, 6720, 17520],
     [1, -504, -16632, -122976, -532728]]
% = [0, 1, -24, 252, -1472, 4830, -6048]
% = 1
```

Modular Form Leaves II

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apply(x->mfcoefs(x,4),[E4,E6])
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     [1, -504, -16632, -122976, -532728]]
% = [0, 1, -24, 252, -1472, 4830, -6048]
% = 1
```

Modular Form Leaves III

```
F = mffrometaquo([1,2;11,2]); mfcoefs(F,10)
G = mffromell(ellinit("11a1"))[2];
mfisequal(F, G)
```

Here `mfetaquo` represents an eta quotient, here $\eta(\tau)^2\eta(11\tau)^2$. The corresponding modular form is equal to the modular form associated to the elliptic curve “11a1” of conductor 11.

```
% = [0, 1, -2, -1, 2, 1, 2, -2, 0, -2, -2]
% = 1
```

Modular Form Leaves III

```
F = mffrometaquo([1,2;11,2]); mfcoefs(F,10)
G = mffromell(ellinit("11a1"))[2];
mfisequal(F, G)
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Here `mfetaquo` represents an eta quotient, here $\eta(\tau)^2\eta(11\tau)^2$. The corresponding modular form is equal to the modular form associated to the elliptic curve “11a1” of conductor 11.

```
% = [0, 1, -2, -1, 2, 1, 2, -2, 0, -2, -2]
% = 1
```

Modular Form Spaces I

```
mf = mfinit([1,12]); L = mfbasis(mf); #L
mfdim(mf)
mfcoefs(L[1],6)
mfcoefs(L[2],6)
```

The default is to ask for the full space $M_k(\Gamma_0(N),\chi)$ (`flag = 4`).

```
% = 2
% = 2
% = [691/65520, 1, 2049, 177148, 4196353, 48828126]
% = [0, 1, -24, 252, -1472, 4830, -6048]
```

Note: for now, the Eisenstein series are given before the cusp forms, and they are normalized with $a(1)=1$, not $a(0)=1$ (which is impossible in general).

Modular Form Spaces I

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```

Note: for now, the Eisenstein series are given before the cusp forms, and they are normalized with $a(1) = 1$, not $a(0) = 1$ (which is impossible in general).

Modular Form Spaces II

Note the direct command

```
mfcoefs(mf,6)
```

which outputs

```
% =
[691/65520     0]
[        1     1]
[     2049   -24]
[   177148   252]
[  4196353 -1472]
[ 48828126  4830]
[362976252 -6048]
```

This command is in general much faster than asking for each individual expansion in the basis.

The cuspidal space is with `flag = 1`:

```
mf = mfinit([1,12], 1); L = mfbasis(mf); #L
mfcoefs(L[1],6)
```

```
% = 1
% = [0, 1, -24, 252, -1472, 4830, -6048]
```

Modular Form Spaces III

The cuspidal space is with `flag = 1`:

```
mf = mfinit([1,12], 1); L = mfbasis(mf); #L
mfcoefs(L[1],6)
```

```
% = 1
% = [0, 1, -24, 252, -1472, 4830, -6048]
```

Modular Form Spaces IV

The newspace is with `flag = 0`:

```
mf = mfinit([35,2], 0); L = mfbasis(mf); #L
for (i = 1, 3, print(mfcoefs(L[i], 10)))
```

(or more simply `mfcoefs(mf,10)` which gives a matrix)

```
% = 3
[0, 3, -1, 0, 3, 1, -8, -1, -9, 1, -1]
[0, -1, 9, -8, -11, -1, 4, 1, 13, 7, 9]
[0, 0, -8, 10, 4, -2, 4, 2, -4, -12, -8]
```

These are (essentially) random modular cusp forms. Usually, one wants eigenforms: this is obtained by the command `mfeigenbasis`, which applies only to the newspace, even if the input is larger:

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(or more simply `mfcoefs(mf,10)` which gives a matrix)

```
% = 3
[0, 3, -1, 0, 3, 1, -8, -1, -9, 1, -1]
[0, -1, 9, -8, -11, -1, 4, 1, 13, 7, 9]
[0, 0, -8, 10, 4, -2, 4, 2, -4, -12, -8]
```

These are (essentially) random modular cusp forms. Usually, one wants eigenforms: this is obtained by the command `mfeigenbasis`, which applies only to the newspace, even if the input is larger:

Modular Form Spaces V

```
mffields(mf)
L = mfeigenbasis(mf); #L
mfcoefs(L[1],10)
mfcoefs(L[2],3)
lift(mfcoefs(L[2],9))
```

```
% = [y, y^2 - y - 4]
% = 2
% = [0, 1, 0, 1, -2, -1, 0, 1, 0, -2, 0]
% = [Mod(0, y^2 - y - 4), Mod(1, y^2 - y - 4),
    Mod(-y, y^2 - y - 4), Mod(y - 1, y^2 - y - 4)]
% = [0, 1, -y, y - 1, y + 2, 1, -4, -1, -y - 4, -y + 2]
```

Modular Form Spaces V

```
mffields(mf)
L = mfeigenbasis(mf); #L
mfcoefs(L[1],10)
mfcoefs(L[2],3)
lift(mfcoefs(L[2],9))
```

```
% = [y, y^2 - y - 4]
% = 2
% = [0, 1, 0, 1, -2, -1, 0, 1, 0, -2, 0]
% = [Mod(0, y^2 - y - 4), Mod(1, y^2 - y - 4),
     Mod(-y, y^2 - y - 4), Mod(y - 1, y^2 - y - 4)]
% = [0, 1, -y, y - 1, y + 2, 1, -4, -1, -y - 4, -y + 2]
```

Very often, need numerical values of coefficients: need to embed in $\mathbb{C}$, so a given eigenform can give several forms. Numerical functions applied to modular forms (for example `mfeval`, which evaluates numerically a form) automatically give a vector of results when there are several embeddings.

To compute the numerical expansion of a form having several embeddings, we use `mfembed` as follows:

```
mfcoefsembed(F,n)=mfembed(F,mfcoefs(F,n));
```

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To compute the numerical expansion of a form having several embeddings, we use `mfembed` as follows:

```
mfcoefsembed(F,n)=mfembed(F,mfcoefs(F,n));
```

Modular Form Spaces VII

We apply to our above example:

```
[V1,V2]=mfcoefsembed(L[2],5);
V1
V2
```

```
% = [0, 1, 1.5615528128088302749107049279870385126,
          -2.5615528128088302749107049279870385126,
           0.43844718719116972508929507201296148743, 1]
% = [0, 1, -2.5615528128088302749107049279870385126,
            1.5615528128088302749107049279870385126,
            4.5615528128088302749107049279870385126, 1]
```

(imaginary parts of $0.E-38$ omitted).

Modular Form Spaces VII

We apply to our above example:

```
[V1,V2]=mfcoefsembed(L[2],5);
V1
V2
```

```
% = [0, 1, 1.5615528128088302749107049279870385126,
         -2.5615528128088302749107049279870385126,
          0.43844718719116972508929507201296148743, 1]
% = [0, 1, -2.5615528128088302749107049279870385126,
           1.5615528128088302749107049279870385126,
           4.5615528128088302749107049279870385126, 1]
```

(imaginary parts of $0.E-38$ omitted).

Modular Form Spaces VIII

Recall:

```
mf = mfinit([35,2], 0); L = mfeigenbasis(mf);
```

```
[mf,F,co] = mffromell(ellinit("35a1")); mfcoefs(F, 10)
mfisequal(F, L[1])
```

```
% = [0, 1, 0, 1, -2, -1, 0, 1, 0, -2, 0]
% = 1
```

```
apply(x->mfdim([96, 2], x), [0..4])
```

```
% = [2, 9, 7, 15, 24]
```

Modular Form Spaces VIII

Recall:

```
mf = mfinit([35,2], 0); L = mfeigenbasis(mf);
```

```
[mf,F,co] = mffromell(ellinit("35a1")); mfcoefs(F, 10)
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```

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% = [0, 1, 0, 1, -2, -1, 0, 1, 0, -2, 0]
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```
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```
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Recall:

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[mf,F,co] = mffromell(ellinit("35a1")); mfcoefs(F, 10)
mfisequal(F, L[1])
```

```
% = [0, 1, 0, 1, -2, -1, 0, 1, 0, -2, 0]
% = 1
```

```
apply(x->mfdim([96, 2], x), [0..4])
```

```
% = [2, 9, 7, 15, 24]
```

Modular Form Spaces VIII

Recall:

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mf = mfinit([35,2], 0); L = mfeigenbasis(mf);
```

```
[mf,F,co] = mffromell(ellinit("35a1")); mfcoefs(F, 10)
mfisequal(F, L[1])
```

```
% = [0, 1, 0, 1, -2, -1, 0, 1, 0, -2, 0]
% = 1
```

```
apply(x->mfdim([96, 2], x), [0..4])
```

```
% = [2, 9, 7, 15, 24]
```

Spaces with Characters

```
mf = mfinit([35,2,5],0); mffields(mf)
F = mfeigenbasis(mf)[1]; lift(mfcoefs(F, 10))
```

Here 5 represents the Legendre–Kronecker symbol $(5/d)$.

```
% = [y^2 + 1]
% = [0, 1, 2*y, -y, -2, -y - 2, 2, -y, 0, 2, -4*y + 2]
```

Because `mffields` gives $y^2 + 1$, in the last output y is equal to one of the two roots of $y^2 + 1 = 0$.
General Dirichlet characters (given in any format) are of course supported.

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Because `mffields` gives $y^2 + 1$, in the last output y is equal to one of the two roots of $y^2 + 1 = 0$.
General Dirichlet characters (given in any format) are of course supported.

```
G = znstar(23, 1);
L = [[G,chi]|chi<-chargalois(G),zncharisodd(G,chi)]; #L
apply(x->mfdim([23,1,x], 1), L)
apply(x->charorder(x[1],x[2]), L)
```

The above shows the most general way to define a Dirichlet character: first define the group G using `znstar(N,1)` (flag 1 necessary), then specify `chi` on generators, e.g., using `chargalois` or otherwise.

```
% = 2
% = [0, 1]
% = [22, 2]
```

Modular Forms of Weight One I

```
G = znstar(23, 1);
L = [[G,chi]|chi<-chargalois(G),zncharisodd(G,chi)]; #L
apply(x->mfdim([23,1,x], 1), L)
apply(x->charorder(x[1],x[2]), L)
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The above shows the most general way to define a Dirichlet character: first define the group G using `znstar(N,1)` (flag 1 necessary), then specify `chi` on generators, e.g., using `chargalois` or otherwise.

```
% = 2
% = [0, 1]
% = [22, 2]
```

```
mfa = mfinit([23,1,0], 1); #mfa
mf = mfa[1]; mfdim(mf)
mfparams(mf)
```

This illustrates wildcards: the 0 (which is of course not limited to weight 1) means that the result is a vector of `mf` of all spaces with given level and weight, but varying character (here, `mfparams` says that the only one is $(-23/n)$).

```
% = 1
% = 1
% = [23, 1, -23, 1]
```

```
mfa = mfinit([23,1,0], 1); #mfa
mf = mfa[1]; mfdim(mf)
mfparams(mf)
```

This illustrates wildcards: the 0 (which is of course not limited to weight 1) means that the result is a vector of `mf` of all spaces with given level and weight, but varying character (here, `mfparams` says that the only one is $(-23/n)$).

```
% = 1
% = 1
% = [23, 1, -23, 1]
```

Here is a little GP script which explores modular forms of weight 1:

```
wt1exp(lim1,lim2)=
{
  my(mfall,mf,chi);
  for(N=lim1,lim2,
    mfall=mfinit([N,1,0], 0); /* Use wildcard */
    for(i=1,#mfall,
      mf=mfall[i];
      chi=mfparams(mf)[3]; /* nice format: D or Mod(a,N) */
      [ print([N,chi,-t]) | t<-mfgaloistype(mf), t < 0 ]
    )
  );
}
```

Modular Forms of Weight One IV

Copy the preceding program from the GP file available with the tutorial on the website: it explores "exotic" weight 1 forms between given levels, i.e., those whose projective image is not dihedral, so cannot easily be constructed explicitly (image A_4 code -12, S_4 code -24, A_5 code -60, opposite of their cardinality).

For instance, try `w1exp(1,230)`, or `w1exp(633,633)`. The latter outputs

```
[633, Mod(71, 633), 2, 10, 60]
```

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For instance, try `w1exp(1,230)`, or `w1exp(633,633)`. The latter outputs

```
[633, Mod(71, 633), 2, 10, 60]
```

These are fully supported, including Hecke operators $T(p^2)$, Cohen–Hurwitz Eisenstein series H_k, Shimura lifts, the Kohnen $+$-space and new space. Simple examples (not using these advanced notions):

```
F = mffrometaquo([2,5;1,-2;4,-2]); Ser(mfcoefs(F,10),q)
T = mfTheta(); mfisequal(F,T)
F = mffromqf(2*matid(3))[2]; Ser(mfcoefs(F,5),q)
mfisequal(F,mfpow(T,3))
```

The first two commands check that $\theta(\tau)=\eta^5(2\tau)/(\eta^2(\tau)\eta^2(4\tau))$.

```
% = 1 + 2*q + 2*q^4 + 2*q^9 + O(q^11)
% = 1
% = 1 + 6*q + 12*q^2 + 8*q^3 + 6*q^4 + 24*q^5 + O(q^6)
% = 1
```

These are fully supported, including Hecke operators $T(p^2)$, Cohen–Hurwitz Eisenstein series H_k, Shimura lifts, the Kohnen $+$-space and new space. Simple examples (not using these advanced notions):

```
F = mffrometaquo([2,5;1,-2;4,-2]); Ser(mfcoefs(F,10),q)
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mfisequal(F,mfpow(T,3))
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The first two commands check that $\theta(\tau) = \eta^5(2\tau)/(\eta^2(\tau)\eta^2(4\tau))$.

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% = 1
% = 1 + 6*q + 12*q^2 + 8*q^3 + 6*q^4 + 24*q^5 + O(q^6)
% = 1
```

Miscellaneous Commands I

```
mf=mfinit([96,6],0); mffields(mf)
mfatkineigenvalues(mf,3)
mf=mfinit([96,3,-3],0); mffields(mf)
mfatkineigenvalues(mf,32)
mfatkineigenvalues(mf,3)
```

```
% = [y, y, y, y, y, y, y^2 - 31, y^2 - 31]
% = [[-1], [-1], [-1], [1], [1], [1], [-1, -1], [1, 1]]
% = [y^4 + 8*y^2 + 9, y^4 + 4*y^2 + 1]
% = [[I, -I, -I, I], [-I, I, I, -I]]
% = [[0.47.... ]] /* complicated complex numbers */
```

The reason we obtain complicated complex numbers in the last command is that the character $(-3/.)$ is not defined modulo $N/Q = 96/3 = 32$. These numbers, called *pseudo-eigenvalues*, are algebraic and of modulus 1.

Miscellaneous Commands I

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mf=mfinit([96,6],0); mffields(mf)
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mf=mfinit([96,3,-3],0); mffields(mf)
mfatkineigenvalues(mf,32)
mfatkineigenvalues(mf,3)
```

```
% = [y, y, y, y, y, y, y^2 - 31, y^2 - 31]
% = [[-1], [-1], [-1], [1], [1], [1], [-1, -1], [1, 1]]
% = [y^4 + 8*y^2 + 9, y^4 + 4*y^2 + 1]
% = [[I, -I, -I, I], [-I, I, I, -I]]
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```

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% = [y, y, y, y, y, y, y^2 - 31, y^2 - 31]
% = [[-1], [-1], [-1], [1], [1], [1], [-1, -1], [1, 1]]
% = [y^4 + 8*y^2 + 9, y^4 + 4*y^2 + 1]
% = [[I, -I, -I, I], [-I, I, I, -I]]
% = [[0.47.... ]] /* complicated complex numbers */
```

The reason we obtain complicated complex numbers in the last command is that the character $(-3/.)$ is not defined modulo $N/Q = 96/3 = 32$. These numbers, called *pseudo-eigenvalues*, are algebraic and of modulus 1.

Miscellaneous Commands II

```
mf = mfinit([96,2]); L = mfbasis(mf);
mfdim([96,2],3)
apply(x->mfconductor(mf,x), L)
```

```
% = 15
% = [16, 32, 48, 96, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96,
     24, 48, 96, 32, 96, 48, 96, 96, 96]
```

Since the dimension of the Eisenstein space (code 3) is 15, this gives the conductors (lowest possible level) of the 15 Eisenstein series, then those of the 9 cusp forms in the given basis of `mf`.

Miscellaneous Commands II

```
mf = mfinit([96,2]); L = mfbasis(mf);
mfdim([96,2],3)
apply(x->mfconductor(mf,x), L)
```

```
% = 15
% = [16, 32, 48, 96, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96,
     24, 48, 96, 32, 96, 48, 96, 96, 96]
```

Since the dimension of the Eisenstein space (code 3) is 15, this gives the conductors (lowest possible level) of the 15 Eisenstein series, then those of the 9 cusp forms in the given basis of `mf`.

Miscellaneous Commands III

```
C = mfcusps(108)
apply(x->mfcuspwidth(108,x), C)
NK = [108,3,-4];
apply(x->mfcuspisregular(NK,x), C)
[c | c<-C, !mfcuspisregular(NK,c)]
```

```
% = [0, 1/2, 1/3, 2/3, 1/4, 1/6, 5/6, 1/9, 2/9, 1/12,
       5/12, 1/18, 5/18, 1/27, 1/36, 5/36, 1/54, 1/108]
% = [108, 27, 12, 12, 27, 3, 3, 4, 4, 3,
        3, 1, 1, 4, 1, 1, 1, 1]
% = [1, 0, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 0, 1]
% = [1/2, 1/6, 5/6, 1/18, 5/18, 1/54]
```

```
C = mfcusps(108)
apply(x->mfcuspwidth(108,x), C)
NK = [108,3,-4];
apply(x->mfcuspisregular(NK,x), C)
[c | c<-C, !mfcuspisregular(NK,c)]
```

```
% = [0, 1/2, 1/3, 2/3, 1/4, 1/6, 5/6, 1/9, 2/9, 1/12,
       5/12, 1/18, 5/18, 1/27, 1/36, 5/36, 1/54, 1/108]
% = [108, 27, 12, 12, 27, 3, 3, 4, 4, 3,
        3, 1, 1, 4, 1, 1, 1, 1]
% = [1, 0, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 0, 1]
% = [1/2, 1/6, 5/6, 1/18, 5/18, 1/54]
```

Miscellaneous Commands IV

```
E4 = mfEk(4); G = mfderivE2(E4); mfcoefs(G, 6)
mfcoefs(mfEk(6), 6)/(-3)
F = mfderivE2(E4, 3); (-9)*mfcoefs(F, 5)
mfisequal(mfEk(10), mflinear([F],[-9]))
```

```
% = [-1/3, 168, 5544, 40992, 177576, 525168, 1352736]
% = [-1/3, 168, 5544, 40992, 177576, 525168, 1352736]
% = [1, -264, -135432, -5196576, -69341448, -515625264]
% = 1
```

```
E4 = mfEk(4); mfeval(mfinit(E4),E4,I)
3*gamma(1/4)^8/(2*Pi)^6
```

```
% = 1.4557628922687093224624220035988692874
% = 1.4557628922687093224624220035988692874
```

```
E4 = mfEk(4); G = mfderivE2(E4); mfcoefs(G, 6)
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Miscellaneous Commands IV

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E4 = mfEk(4); mfeval(mfinit(E4),E4,I)
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```

Miscellaneous Commands IV

```
E4 = mfEk(4); G = mfderivE2(E4); mfcoefs(G, 6)
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mfisequal(mfEk(10), mflinear([F],[-9]))
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% = [-1/3, 168, 5544, 40992, 177576, 525168, 1352736]
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% = [1, -264, -135432, -5196576, -69341448, -515625264]
% = 1
```

```
E4 = mfEk(4); mfeval(mfinit(E4),E4,I)
3*gamma(1/4)^8/(2*Pi)^6
```

```
% = 1.4557628922687093224624220035988692874
% = 1.4557628922687093224624220035988692874
```

Miscellaneous Commands V

```
mf = mfinit([96,4], 0); M = mfheckemat(mf, 7)
```

```
% =
[0     0    0     372     696    0]

[0     0   36       0       0  -96]

[0  27/5    0  -276/5  -276/5    0]

[1     0  -12       0       0   62]

[0     0    1       0       0  -16]

[0  -3/5    0    14/5   -16/5    0]
```

Miscellaneous Commands V

```
mf = mfinit([96,4], 0); M = mfheckemat(mf, 7)
```

```
% =
[0     0   0    372    696   0]

[0     0  36      0      0 -96]

[0 27/5   0 -276/5 -276/5   0]

[1     0 -12      0      0  62]

[0     0   1      0      0 -16]

[0 -3/5   0   14/5  -16/5   0]
```

Miscellaneous Commands VI

```
P = charpoly(M)
print(factor(P))
```

```
% = x^6 - 1456*x^4 + 209664*x^2 - 2985984
[x - 36, 1; x - 12, 1; x - 4, 1; x + 4, 1;
                       x + 12, 1; x + 36, 1]
```

Note that this shows that all the eigenvalues of $T(7)$ are integral, so the splitting will be entirely rational and the eigenforms with integral coefficients. Let's check:

Miscellaneous Commands VI

```
P = charpoly(M)
print(factor(P))
```

```
% = x^6 - 1456*x^4 + 209664*x^2 - 2985984
[x - 36, 1; x - 12, 1; x - 4, 1; x + 4, 1;
                      x + 12, 1; x + 36, 1]
```

Note that this shows that all the eigenvalues of $T(7)$ are integral, so the splitting will be entirely rational and the eigenforms with integral coefficients. Let's check:

Miscellaneous Commands VII

```
mffields(mf)
L = mfeigenbasis(mf); for(i=1,6,print(mfcoefs(L[i],15)))
```

```
% = [y, y, y, y, y, y]
[0, 1, 0, 3, 0, 10, 0, 4, 0, 9, 0, -20, 0, 70, 0, 30]
[0, 1, 0, 3, 0, 2, 0, 12, 0, 9, 0, 60, 0, -42, 0, 6]
[0, 1, 0, 3, 0, -14, 0, -36, 0, 9, 0, -36, 0, 54, 0, -42]
[0, 1, 0, -3, 0, 10, 0, -4, 0, 9, 0, 20, 0, 70, 0, -30]
[0, 1, 0, -3, 0, 2, 0, -12, 0, 9, 0, -60, 0, -42, 0, -6]
[0, 1, 0, -3, 0, -14, 0, 36, 0, 9, 0, 36, 0, 54, 0, 42]
```

Note again the twisting phenomenon: there are three eigenforms, and three twists by the character $(-4/n)$.

Miscellaneous Commands VII

```
mffields(mf)
L = mfeigenbasis(mf); for(i=1,6,print(mfcoefs(L[i],15)))
```

```
% = [y, y, y, y, y, y]
[0, 1, 0, 3, 0, 10, 0, 4, 0, 9, 0, -20, 0, 70, 0, 30]
[0, 1, 0, 3, 0, 2, 0, 12, 0, 9, 0, 60, 0, -42, 0, 6]
[0, 1, 0, 3, 0, -14, 0, -36, 0, 9, 0, -36, 0, 54, 0, -42]
[0, 1, 0, -3, 0, 10, 0, -4, 0, 9, 0, 20, 0, 70, 0, -30]
[0, 1, 0, -3, 0, 2, 0, -12, 0, 9, 0, -60, 0, -42, 0, -6]
[0, 1, 0, -3, 0, -14, 0, 36, 0, 9, 0, 36, 0, 54, 0, 42]
```

Note again the twisting phenomenon: there are three eigenforms, and three twists by the character $(-4/n)$.

Miscellaneous Commands VIII

```
[mfB,M,C]=mfatkininit(mf,3); M
```

```
% =
[   0 -3    0    0  -24    0]

[-1/3  0 -4/3    0    0  -12]

[   0  0    0 -9/5 -6/5    0]

[   0  0 -2/3    0    0   -1]

[   0  0  1/6    0    0  3/2]

[   0  0    0  1/5  4/5    0]
```

Miscellaneous Commands VIII

```
[mfB,M,C]=mfatkininit(mf,3); M
```

```
% =
[    0 -3     0     0   -24    0]

[-1/3  0 -4/3     0     0  -12]

[    0  0     0 -9/5 -6/5    0]

[    0  0 -2/3     0     0   -1]

[    0  0  1/6     0     0  3/2]

[    0  0     0  1/5   4/5    0]
```

The matrix of the Atkin–Lehner involution W_Q is the above matrix divided by C, but here $C = 1$: `[C,matdet(M/C)]` outputs $[1, -1]$. Since the eigenvalues are real in even weight and no character, this means that there is an odd number of -1, hence an odd number of $+1$:

```
[C,matdet(M/C)]
mfatkineigenvalues(mf,3)
```

```
% = [1, -1]
% = [[-1], [-1], [-1], [1], [1], [1]]
```

Miscellaneous Commands IX

The matrix of the Atkin–Lehner involution W_Q is the above matrix divided by C, but here $C = 1$: `[C,matdet(M/C)]` outputs $[1, -1]$. Since the eigenvalues are real in even weight and no character, this means that there is an odd number of -1, hence an odd number of $+1$:

```
[C,matdet(M/C)]
mfatkineigenvalues(mf,3)
```

```
% = [1, -1]
% = [[-1], [-1], [-1], [1], [1], [1]]
```

Combination with L-Functions I

```
E4 = mfEk(4); mf = mfinit(E4); LE = lfunmf(mf, E4);
lfun(LE, 2)/Pi^2
lfun(LE, 0)
D = mfDelta(); mf = mfinit(D); L = lfunmf(mf, D);
lfunlambda(L, 3)/lfunlambda(L, 5)
r = lfunlambda(L, 1)/lfunlambda(L, 3)
bestappr(r)
```

```
% = -3.3333333333333333333333333333333333333
% = -1.0000000000000000000000000000000000000
% = 1.5555555555555555555555555555555555556
% = 2.3444283646888567293777134587554269175
% = 1620/691
```

```
LIN = lfuninit(L, [50]);
ploth(t = 0, 50, lfunhardy(LIN, t))
```

Combination with L-Functions I

```
E4 = mfEk(4); mf = mfinit(E4); LE = lfunmf(mf, E4);
lfun(LE, 2)/Pi^2
lfun(LE, 0)
D = mfDelta(); mf = mfinit(D); L = lfunmf(mf, D);
lfunlambda(L, 3)/lfunlambda(L, 5)
r = lfunlambda(L, 1)/lfunlambda(L, 3)
bestappr(r)
```

```
% = -3.333333333333333333333333333333333333333
% = -1.000000000000000000000000000000000000000
% = 1.555555555555555555555555555555555555556
% = 2.344428364688856729377713458755426917
% = 1620/691
```

```
LIN = lfuninit(L, [50]);
ploth(t = 0, 50, lfunhardy(LIN, t))
```

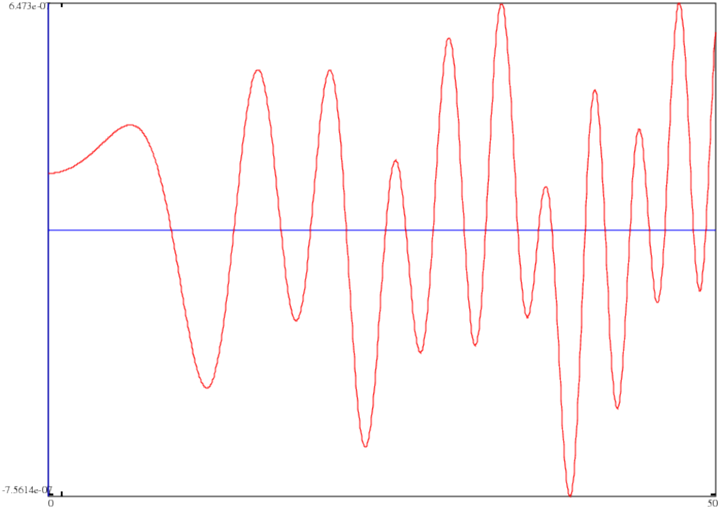
Combination with L-Functions I

```
E4 = mfEk(4); mf = mfinit(E4); LE = lfunmf(mf, E4);
lfun(LE, 2)/Pi^2
lfun(LE, 0)
D = mfDelta(); mf = mfinit(D); L = lfunmf(mf, D);
lfunlambda(L, 3)/lfunlambda(L, 5)
r = lfunlambda(L, 1)/lfunlambda(L, 3)
bestappr(r)
```

```
% = -3.333333333333333333333333333333333333
% = -1.000000000000000000000000000000000000
% = 1.5555555555555555555555555555555555556
% = 2.3444283646888567293777134587554269175
% = 1620/691
```

```
LIN = lfuninit(L, [50]);
ploth(t = 0, 50, lfunhardy(LIN, t))
```

Combination with L-Functions II



Miscellaneous Commands IX

```
E4 = mfEk(4); F = mfbracket(E4, E4, 2); mfcoefs(F, 6)/4800
D = mfDelta(); mftaylor(D, 9)*1728
D3 = mftwist(D, -3); mfcoefs(D3, 9)
P = mfparams(D3)
mf = mfinit(D3, 1); mftobasis(mf, D3)
```

```
% = [0, 1, -24, 252, -1472, 4830, -6048]
% = [1, 0, -1/12, 0, 1/96, 0, 1/288, 0, -11/2304, 0]
% = [0, 1, 24, 0, -1472, -4830, 0, -16744, -84480, 0]
% = [9, 12, 1, y]
% = [0, 0, 0, 0, 0,
     5546/4131, -1232/12393, -47/16524, 11/24786]~
```

Miscellaneous Commands IX

```
E4 = mfEk(4); F = mfbracket(E4, E4, 2); mfcoefs(F, 6)/4800
D = mfDelta(); mftaylor(D, 9)*1728
D3 = mftwist(D, -3); mfcoefs(D3, 9)
P = mfparams(D3)
mf = mfinit(D3, 1); mftobasis(mf, D3)
```

```
% = [0, 1, -24, 252, -1472, 4830, -6048]
% = [1, 0, -1/12, 0, 1/96, 0, 1/288, 0, -11/2304, 0]
% = [0, 1, 24, 0, -1472, -4830, 0, -16744, -84480, 0]
% = [9, 12, 1, y]
% = [0, 0, 0, 0, 0,
     5546/4131, -1232/12393, -47/16524, 11/24786]~
```

Miscellaneous Commands X

```
F = mffromell(ellinit("49a1"))[2]; mfisCM(F)
mfisequal(F, mftwist(F, -7))
mf = mfinit([23,1,-23],1); F = mfeigenbasis(mf)[1];
mfisCM(F)
mfisequal(F, mftwist(F, -23))
```

```
% = -7
% = 1
% = -23
% = 0
```

Miscellaneous Commands X

```
F = mffromell(ellinit("49a1"))[2]; mfisCM(F)
mfisequal(F, mftwist(F, -7))
mf = mfinit([23,1,-23],1); F = mfeigenbasis(mf)[1];
mfisCM(F)
mfisequal(F, mftwist(F, -23))
```

```
% = -7
% = 1
% = -23
% = 0
```

We want to search for normalized eigenforms with integral (equivalently, rational) Fourier coefficients, given a few $a(p)$ for p prime, possibly modulo something.

```
L = mfeigensearch([[1..30],4], [[2,2],[3,-1]]); #L
F = L[1]; mfparams(F)
mfcoefs(F, 10)
```

```
% = 1
% = [26, 4, 1, y]
% = [0, 1, 2, -1, 4, 17, -2, -35, 8, -26, 34]
```

Searching: `mfeigensearch` I

We want to search for normalized eigenforms with integral (equivalently, rational) Fourier coefficients, given a few $a(p)$ for p prime, possibly modulo something.

```
L = mfeigensearch([[1..30],4], [[2,2],[3,-1]]); #L
F = L[1]; mfparams(F)
mfcoefs(F, 10)
```

```
% = 1
% = [26, 4, 1, y]
% = [0, 1, 2, -1, 4, 17, -2, -35, 8, -26, 34]
```

```
L=mfeigensearch([[1..30],4], [[2,Mod(2,5)], [3,Mod(-1,5)]])
[ mfparams(F)[1] | F <- L ]
F1 = L[1]; mfcoefs(F1, 10)
F2 = L[2]; mfcoefs(F2, 10)
F = mflinear([F1, F2], [-1, 1]); mfcoefs(F, 14)/5
mfsturm([26,4])
```

```
% = [26, 26]
% = [0, 1, 2, -1, 4, 17, -2, -35, 8, -26, 34]
% = [0, 1, 2, 4, 4, -18, 8, 20, 8, -11, -36]
% = [0, 0, 0, 1, 0, -7, 2, 11, 0, 3, -14, -10, 4, 0, 22]
% = 15
```

Searching: mfeigensearch II

```
L=mfeigensearch([[1..30],4], [[2,Mod(2,5)], [3,Mod(-1,5)]])
[ mfparams(F)[1] | F <- L ]
F1 = L[1]; mfcoefs(F1, 10)
F2 = L[2]; mfcoefs(F2, 10)
F = mflinear([F1, F2], [-1, 1]); mfcoefs(F, 14)/5
mfsturm([26,4])
```

```
% = [26, 26]
% = [0, 1, 2, -1, 4, 17, -2, -35, 8, -26, 34]
% = [0, 1, 2, 4, 4, -18, 8, 20, 8, -11, -36]
% = [0, 0, 0, 1, 0, -7, 2, 11, 0, 3, -14, -10, 4, 0, 22]
% = 15
```

Searching: mfsearch

A more primitive searching is the mfsearch command:

```
W = mfsearch([[1..35],3],[0,1,2,3,4,5,6,7,8],1);
[ mfparams(F) | F <- W]
mfcoefs(W[1],10)
mfcoefs(W[2],10)
```

```
% = [[30, 3, -3, y], [30, 3, -15, y]]
% = [0, 1, 2, 3, 4, 5, 6, 7, 8, -14, -30]
% = [0, 1, 2, 3, 4, 5, 6, 7, 8, -21, -50]
```

We are searching for modular forms with rational coefficients, of weight 3 and level less than or equal to 35, in the cuspidal space (code 1) whose Fourier expansion begins with $q + 2q^2 + 3q^3 + 4q^4 + 5q^5 + 6q^6 + 7q^7 + 8q^8 + \cdots$. We find that there are two, both of level 30, one with character $(-3/.)$, the second $(-15/.)$, and we give 11 coefficients.

Searching: mfsearch

A more primitive searching is the mfsearch command:

```
W = mfsearch([[1..35],3],[0,1,2,3,4,5,6,7,8],1);
[ mfparams(F) | F <- W]
mfcoefs(W[1],10)
mfcoefs(W[2],10)
```

```
% = [[30, 3, -3, y], [30, 3, -15, y]]
% = [0, 1, 2, 3, 4, 5, 6, 7, 8, -14, -30]
% = [0, 1, 2, 3, 4, 5, 6, 7, 8, -21, -50]
```

We are searching for modular forms with rational coefficients, of weight 3 and level less than or equal to 35, in the cuspidal space (code 1) whose Fourier expansion begins with $q+2q^2+3q^3+4q^4+5q^5+6q^6+7q^7+8q^8+\cdots$. We find that there are two, both of level 30, one with character $(-3/.)$, the second $(-15/.)$, and we give 11 coefficients.

Advanced Commands

The `Pari/GP` modular form package is unique in that it implements a number of advanced functions on modular forms not available in other packages:

1. Fourier expansion of $F|_k\gamma$, and in particular expansion at any cusp.
2. Numerical computation of Atkin–Lehner pseudo-eigenvalues.
3. Numerical evaluation of a form near the real axis.
4. Numerical computation of symbols, i.e., integrals over any path.
5. Numerical computation of general Petersson products.

This is based on the computation of bases of modular form spaces made of products of Eisenstein series, and of general expansions of these series. Although more expensive than previous computations, once the precomputations are done the rest is essentially immediate. In practice levels up to 500 are reachable in reasonable weight.

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5. Numerical computation of general Petersson products.

This is based on the computation of bases of modular form spaces made of products of Eisenstein series, and of general expansions of these series. Although more expensive than previous computations, once the precomputations are done the rest is essentially immediate. In practice levels up to 500 are reachable in reasonable weight.

```
mf = mfinit([32,4],0); F = mfbasis(mf)[1]; mfcoefs(F,10)
mfslashexpansion(mf,F,[0,-1;32,0],10,1,&A);
A
```

Here we ask for the action of the Fricke involution $\tau \mapsto -1/(32\tau)$ on F; the parameter 1 asks the program to "rationalize" the result, and A will be explained below.

```
% = [0, 3, 0, 0, 0, 2, 0, 0, 0, 47, 0]
% = [0, 1, 0, 16, 0, 22, 0, 32, 0, -27, 0]
% = [0, 1]
```

$A=[0,1]$ means that the expansion will be of the form $q^0\sum_{n\ge0}a(n)q^{n/1}$, here simply $\sum_{n\ge0}a(n)q^n$. Thus

$$F|_4W_{32}=q+16q^3+22q^5+32q^7-27q^9+O(q^{11})\ .$$

Fourier expansion of $F|_k\gamma$ I

```
mf = mfinit([32,4],0); F = mfbasis(mf)[1]; mfcoefs(F,10)
mfslashexpansion(mf,F,[0,-1;32,0],10,1,&A);
A
```

Here we ask for the action of the Fricke involution $\tau \mapsto -1/(32\tau)$ on F; the parameter 1 asks the program to "rationalize" the result, and A will be explained below.

```
% = [0, 3, 0, 0, 0, 2, 0, 0, 0, 47, 0]
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$$F|_4 W_{32} = q + 16q^3 + 22q^5 + 32q^7 - 27q^9 + O(q^{11})\,.$$

```
mf = mfinit([12,8],0); F = mfbasis(mf)[1];
mfslashexpansion(mf,F,[1,0;2,1],7,0,&A)
A
mfslashexpansion(mf,F,[1,0;2,1],7,1,&A)
```

```
% = [0, 0, 0, 0.6666666... + 0.E-38*I, 0,
            -3.99999999... + 6.9282032302...*I, 0,
           -11.99999999... - 20.7846096908...*I]
% = [0, 3]
% = [0, 0, 0, 2/3, 0, Mod(8*t, t^2 + t + 1),
      0, Mod(-24*t - 24, t^2 + t + 1)]
```

Here $A = [0,3]$ so the expansion is in powers of $q^{1/3}$ (still with q^0 in front); the first command (parameter 0) gives the coefficients as complex numbers (whose real part is easy to recognize), and the last (parameter 1) “rationalizes” the result, showing that these coefficients seem to be (are in fact) in $\mathbb{Q}(\exp(2\pi i/3))$.

Fourier expansion of $F|_k\gamma$ II

```
mf = mfinit([12,8],0); F = mfbasis(mf)[1];
mfslashexpansion(mf,F,[1,0;2,1],7,0,&A)
A
mfslashexpansion(mf,F,[1,0;2,1],7,1,&A)
```

```
% = [0, 0, 0, 0.6666666... + 0.E-38*I, 0,
            -3.99999999... + 6.9282032302...*I, 0,
           -11.99999999... - 20.7846096908...*I]
% = [0, 3]
% = [0, 0, 0, 2/3, 0, Mod(8*t, t^2 + t + 1),
      0, Mod(-24*t - 24, t^2 + t + 1)]
```

Here $A = [0, 3]$ so the expansion is in powers of $q^{1/3}$ (still with q^0 in front); the first command (parameter 0) gives the coefficients as complex numbers (whose real part is easy to recognize), and the last (parameter 1) "rationalizes" the result, showing that these coefficients seem to be (are in fact) in $\mathbb{Q}(\exp(2\pi i/3))$.

Fourier expansion of $F|_k\gamma$ III

```
mf = mfinit([12,7,-4],0); F = mfbasis(mf)[1];
mfslashexpansion(mf,F,[1,0;6,1],5,1,&A)
A
```

```
% = [-5/32, 81/32, 21/16, -597/8, 1215/32, 1689/8]
% = [1/2, 1]
```

Here we have an example with $A[1] = 1/2 \neq 0$: we have

$$F|_7\begin{pmatrix}1&0\\6&1\end{pmatrix} = q^{1/2}(-5/32+(81/32)q+(21/16)q^2-(597/8)q^3+\cdots)\ .$$

Fourier expansion of $F|_k\gamma$ III

```
mf = mfinit([12,7,-4],0); F = mfbasis(mf)[1];
mfslashexpansion(mf,F,[1,0;6,1],5,1,&A)
A
```

```
% = [-5/32, 81/32, 21/16, -597/8, 1215/32, 1689/8]
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Here we have an example with $A[1] = 1/2 \neq 0$: we have

$$F|_7 \left(\begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix}\right) = q^{1/2}(-5/32+(81/32)q+(21/16)q^2-(597/8)q^3+\cdots)\,.$$

Evaluation of a Form I

`mfeval` can easily evaluate a form near the real axis:

```
mf = mfinit([12,4],1); F = mfbasis(mf)[1];
mfeval(mf,F,1/Pi+10^(-6)*I)
mfeval(mf,F,1/Pi+10^(-7)*I)
mfeval(mf,F,1/Pi+10^(-8)*I)
```

```
% = -89811.049350396250531782882568405506024
    - 58409.940965200894541585402642924371696*I
% = 4.821246850466111318325339669181329261 E-52
   + 6.7885262281520647908871247541561415340 E-52*I
% = 0
```

These results are immediate and correct: at height 10^{-6} the value is large, at height 10^{-7} very small (and really of the order of 10^{-52} with 30 correct decimals). Of course the value is not exactly 0 at height 10^{-8} but cannot be computed with 38 decimals default accuracy (simply increase the accuracy to $57D$, the value is of the order of 10^{-69}).

Evaluation of a Form I

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mf = mfinit([12,4],1); F = mfbasis(mf)[1];
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mfeval(mf,F,1/Pi+10^(-7)*I)
mfeval(mf,F,1/Pi+10^(-8)*I)
```

```
% = -89811.04935039625053178288256840550602 4
    - 58409.94096520089454158540264292437169 6*I
% = 4.821246850466111318325339669181329226 1 E-52
   + 6.788526228152064790887124754156141534 0 E-52*I
% = 0
```

These results are immediate and correct: at height 10^{-6} the value is large, at height 10^{-7} very small (and really of the order of 10^{-52} with 30 correct decimals). Of course the value is not exactly 0 at height 10^{-8} but cannot be computed with 38 decimals default accuracy (simply increase the accuracy to $57D$, the value is of the order of 10^{-69}).

Evaluation of a Form II

Second, it can also evaluate forms at cusps:

```
T = mfTheta(); mf = mfinit(T); mfeval(mf,T,[0,1/2,1,oo])
```

```
% = [1/2 - 1/2*I, 0, 1/2 - 1/2*I, 1]
```

Warning: the value at a cusp is not the limit as τ tends to the cusp because of the automorphy factor $(c\tau + d)^{-k}$:

```
mfeval(mf,T,10^(-8)*I)
```

```
% = -7071.0678118654752440084436210484903928
       + 2.4074124304840448l6 E-35*I
```

This number is equal to $-10^4\sqrt{2}/2$.

Evaluation of a Form II

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Evaluation of a Form II

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T = mfTheta(); mf = mfinit(T); mfeval(mf,T,[0,1/2,1,oo])
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```
% = [1/2 - 1/2*I, 0, 1/2 - 1/2*I, 1]
```

Warning: the value at a cusp is not the limit as τ tends to the cusp because of the automorphy factor $(c\tau + d)^{-k}$:

```
mfeval(mf,T,10^(-8)*I)
```

```
% = -7071.0678118654752440084436210484903928
      + 2.4074124304840448 16 E-35*I
```

This number is equal to $-10^4\sqrt{2}/2$.

If F has weight $k \geq 2$ integral, a generalized period is the polynomial given by the integral

$$J(F; s_1, s_2) = \int_{s_1}^{s_2} (X - \tau)^{k-2} F(\tau)\, d\tau\ ,$$

where s_i are points in the completed upper-half plane. In particular the coefficients give the integrals of $\tau^j F(\tau)$ for $0 \leq j \leq k - 2$.

Most important when s_i are cusps. Necessary precomputation of symbols (no need for the definition), then other computations immediate. Also necessary for Petersson products.

Periods and Symbols II

```
mf = mfinit([35,2],1); F = mfbasis(mf)[1];
FS = mfsymbol(mf,F);
mfsymboleval(FS,[0,oo])
mfsymboleval(FS,[1/2,3/5])
mfsymboleval(FS,[I,2*I])
mfsymboleval(FS,[1/2,I])
```

```
% =  0.31404011074188471664161704390256378537*I
% = -0.14296962919184795604253140534195291798
    - 0.26199756419561033271653744806924309759*I
% =  0.00088969563028739893631700037491116258378*I
% = -0.61518300331940868645187865843466669894*I
```

Periods and Symbols II

```
mf = mfinit([35,2],1); F = mfbasis(mf)[1];
FS = mfsymbol(mf,F);
mfsymboleval(FS,[0,oo])
mfsymboleval(FS,[1/2,3/5])
mfsymboleval(FS,[I,2*I])
mfsymboleval(FS,[1/2,I])
```

```
% =  0.31404011074188471664161704390256378537*I
% = -0.14296962919184795604253140534195291798
    - 0.26199756419561033271653744806924309759*I
% =  0.00088969563028739893631700037491116258378*I
% = -0.61518300331940868645187865843466669894*I
```

Periods and Symbols III

```
mf = mfinit([5,4],1); F = mfbasis(mf)[1];
FS = mfsymbol(mf,F);
mfsymboleval(FS,[0,oo])
```

```
% = 0.025682886503399670885091327035730701191*I*x^2
   + 0.020865138644297634350206531603632923359*x
   - 0.0051365773006799341770182654071461402382*I
```

Note that `mfsymboleval` can also be applied to noncuspidal forms: in case of divergent integrals the result is a rational function or a polynomial of degree $k-1$, which can easily be interpreted.

Periods and Symbols III

```
mf = mfinit([5,4],1); F = mfbasis(mf)[1];
FS = mfsymbol(mf,F);
mfsymboleval(FS,[0,oo])
```

```
% = 0.025682886503399670885091327035730701191*I*x^2
   + 0.020865138644297634350206531603632923359*x
   - 0.0051365773006799341770182654071461402382*I
```

Note that `mfsymboleval` can also be applied to noncuspidal forms: in case of divergent integrals the result is a rational function or a polynomial of degree $k-1$, which can easily be interpreted.

Periods and Symbols IV

```
T4 = mfpow(mfTheta(),4); mf = mfinit(T4);
TS = mfsymbol(mf,T4);
mfsymboleval(TS,[0,oo])
mfsymboleval(TS,[1/2,oo])
mfsymboleval(TS,[1/2,355/226])
```

```
% = (1.0000000000000000000000000000000000000*x^2
    - 0.88254240061060637358582572847199076393*I*x
    - 0.25000000000000000000000000000000000000)/x
% =  1.0000000000000000000000000000000000000*x
  + (-0.50000000000000000000000000000000000000
    - 0.44127120030530318679291286423599538197*I)
% = -7.0000000000000000000000000000000000000
```

First result: rational function degree 2 / degree 1, divergent integral. Second result: polynomial of degree $1 = k - 1 > k - 2$, divergent integral. Third result: polynomial of degree $0 = k - 2$, convergent integral (prove -7).

Periods and Symbols IV

```
T4 = mfpow(mfTheta(),4); mf = mfinit(T4);
TS = mfsymbol(mf,T4);
mfsymboleval(TS,[0,oo])
mfsymboleval(TS,[1/2,oo])
mfsymboleval(TS,[1/2,355/226])
```

```
% = (1.0000000000000000000000000000000000000*x^2
    - 0.88254240061060637358582572847199076393*I*x
    - 0.25000000000000000000000000000000000000)/x
% =  1.0000000000000000000000000000000000000*x
  + (-0.50000000000000000000000000000000000000
    - 0.44127120030530318679291286423599538197*I)
% = -7.0000000000000000000000000000000000000
```

First result: rational function degree 2 / degree 1, divergent integral. Second result: polynomial of degree $1 = k - 1 > k - 2$, divergent integral. Third result: polynomial of degree $0 = k - 2$, convergent integral (prove -7).

Periods and Symbols IV

```
T4 = mfpow(mfTheta(),4); mf = mfinit(T4);
TS = mfsymbol(mf,T4);
mfsymboleval(TS,[0,oo])
mfsymboleval(TS,[1/2,oo])
mfsymboleval(TS,[1/2,355/226])
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% = (1.0000000000000000000000000000000000000*x^2
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Periods and Symbols IV

```
T4 = mfpow(mfTheta(),4); mf = mfinit(T4);
TS = mfsymbol(mf,T4);
mfsymboleval(TS,[0,oo])
mfsymboleval(TS,[1/2,oo])
mfsymboleval(TS,[1/2,355/226])
```

```
% = (1.0000000000000000000000000000000000000*x^2
    - 0.88254240061060637358582572847199076393*I*x
    - 0.25000000000000000000000000000000000000)/x
% =  1.0000000000000000000000000000000000000*x
  + (-0.50000000000000000000000000000000000000
    - 0.44127120030530318679291286423599538197*I)
% = -7.0000000000000000000000000000000000000
```

First result: rational function degree 2 / degree 1, divergent integral. Second result: polynomial of degree $1 = k - 1 > k - 2$, divergent integral. Third result: polynomial of degree $0 = k - 2$, convergent integral (prove -7).

Periods and Symbols V

There also exist simpler functions `mfperiodpol` (integral from 0 to ∞) and mfperiodpolbasis (only in level 1):

```
# /* timer on */
mf = mfinit([96,6],0); F = mfbasis(mf)[1];
FS = mfsymbol(mf,F);
mfsymboleval(FS,[0,oo]);
mfperiodpol(mf,F);
```

```
time = 24 ms.
time = 9,477 ms.
time = 0 ms.
time = 76 ms.
```

(results on next page).
The `mfsymbol` computation requires 9.477 seconds, but the evaluation is instantaneous. If you only need the integral from 0 to ∞, as here, no need for symbols, the computation requires only 0.076 seconds.

Periods and Symbols V

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```
# /* timer on */
mf = mfinit([96,6],0); F = mfbasis(mf)[1];
FS = mfsymbol(mf,F);
mfsymboleval(FS,[0,oo]);
mfperiodpol(mf,F);
```

```
time = 24 ms.
time = 9,477 ms.
time = 0 ms.
time = 76 ms.
```

(results on next page).
The `mfsymbol` computation requires 9.477 seconds, but the evaluation is instantaneous. If you only need the integral from 0 to ∞, as here, no need for symbols, the computation requires only 0.076 seconds.

Periods and Symbols VI

```
% = 46.366702389191867463049266055452963967*I*x^4
    + 3.8953700388682004473225316269956194525*x^3
    - 0.56826542231980277465186820072941104401*I*x^2
    - 0.15489398386891152199982272551206710377*x
    + 0.024487897732315785610377476118978713061*I
% = /* same result */
```

Petersson Products I

Recall the Petersson product in level N and weight k:

$$< F, G >= \frac{1}{[\Gamma : \Gamma_0(N)]} \int_{\Gamma_0(N)\backslash\mathbb{H}} y^k F(\tau)\overline{G(\tau)} \frac{dxdy}{y^2} \; .$$

This is available for any two forms, even for non eigenforms or noncuspidal, as long as the integral converges; it needs the precomputation of symbols using `mfsymbol`. As usual, this precomputation may take some time, but the subsequent ones are essentially instantaneous.

Petersson Products II

```
mf = mfinit([96,4],0); [F1,F2] = mfbasis(mf);
FS1 = mfsymbol(mf,F1); FS2 = mfsymbol(mf,F2);
mfpetersson(FS1)
mfpetersson(FS2)
mfpetersson(FS1,FS2)
```

```
% = 0.00061471684149817788924091516302517391826
% = 0.0055324515734836010031682364672265652647
% = 1.6262535777977610381 E-40 + 1.2754930021943223828 E-4
```

The `mfsymbol` computations take each 2.5 seconds, but after everything is instantaneous. Note that `mfpetersson(FS,FS)` can be abbreviated to `mfpetersson(FS)`. Also, even though $F1$ and $F2$ are not eigenforms, the last result seem to show that they are orthogonal: this is true, prove it!

Petersson Products II

```
mf = mfinit([96,4],0); [F1,F2] = mfbasis(mf);
FS1 = mfsymbol(mf,F1); FS2 = mfsymbol(mf,F2);
mfpetersson(FS1)
mfpetersson(FS2)
mfpetersson(FS1,FS2)
```

```
% = 0.00061471684149817788924091516302517391826
% = 0.0055324515734836010031682364672265652647
% = 1.6262535777977610381 E-40 + 1.2754930021943223828 E-4
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The `mfsymbol` computations take each 2.5 seconds, but after everything is instantaneous. Note that `mfpetersson(FS,FS)` can be abbreviated to `mfpetersson(FS)`. Also, even though $F1$ and $F2$ are not eigenforms, the last result seem to show that they are orthogonal: this is true, prove it!

Example of noncuspidal Petersson products:

```
mf12 = mfinit([12,5,-3]);
E1 = mfeisenstein(5,1,-3);
E2 = mfeisenstein(5,-3,1);
cusps = mfcusps(12)
[mfcuspval(mf12,E1,c) | c<-cusps]
[mfcuspval(mf12,E2,c) | c<-cusps]
```

```
% = [  0, 1/2, 1/3, 1/4, 1/6, 1/12]
% = [  0,   0,   1,   0,   1,    1]
% = [1/3, 1/3,   0, 1/3,   0,    0]
```

mfcuspval computes the valuation of a form at a cusp. The above results show that at the six cusps of $\Gamma_0(12)$, one of the two Eisenstein series vanishes, so their Petersson product will converge.

Petersson Products III

Example of noncuspidal Petersson products:

```
mf12 = mfinit([12,5,-3]);
E1 = mfeisenstein(5,1,-3);
E2 = mfeisenstein(5,-3,1);
cusps = mfcusps(12)
[mfcuspval(mf12,E1,c) | c<-cusps]
[mfcuspval(mf12,E2,c) | c<-cusps]
```

```
% = [  0, 1/2, 1/3, 1/4, 1/6, 1/12]
% = [  0,   0,   1,   0,   1,    1]
% = [1/3, 1/3,   0, 1/3,   0,    0]
```

`mfcuspval` computes the valuation of a form at a cusp. The above results show that at the six cusps of $\Gamma_0(12)$, one of the two Eisenstein series vanishes, so their Petersson product will converge.

Petersson Products III

```
P(mf) = mfpetersson(mfsymbol(mf,E1),mfsymbol(mf,E2));
mf3 = mfinit([3,5,-3]); mf96 = mfinit([96,5,-3]);
P(mf12)
P(mf3);
P(mf96);
```

```
time = 149 ms.
% = -1.8848216716468969562647734582232071466 E-5
    -1.9057659114817512165 E-43*I
time = 16 ms.
time = 4,412 ms.
```

Of course, because of the normalizing factor $1/[\Gamma : \Gamma_0(N)]$ all results are the same, but the required time increases very fast with the level (at least like its square).

Petersson Products III

```
P(mf) = mfpetersson(mfsymbol(mf,E1),mfsymbol(mf,E2));
mf3 = mfinit([3,5,-3]); mf96 = mfinit([96,5,-3]);
P(mf12)
P(mf3);
P(mf96);
```

```
time = 149 ms.
% = -1.8848216716468969562647734582232071466 E-5
    -1.9057659114817512165 E-43*I
time = 16 ms.
time = 4,412 ms.
```

Of course, because of the normalizing factor $1/[\Gamma:\Gamma_0(N)]$ all results are the same, but the required time increases very fast with the level (at least like its square).

Thank you for your attention !