Sato-Tate Like Problems on Half-Integral Weight Modular Forms

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Basic Definitions

Modular Group

$$\Gamma = SL(2,\mathbb{Z}) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad-bc=1\}.$$

Congruence Subgroup

$$\Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 (\mathrm{mod} N)\}.$$

Modular Form

Let f be a complex-valued function on the upper half plane

$$\mathfrak{U} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

satisfying the following three conditions:

(i) f is a holomorphic on $\mathfrak{U}$,

(ii) for any $z \in \mathfrak{U}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

where k is a positive integer,

(iii) f is holomorphic at ∞.

Then f is called a *modular form of weight k for the modular group* Γ.
A modular form f that vanishes at ∞ is called a *cusp form*.

It is clear that

$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are generators of the modular group and $T(z) := z + 1$ and $S := -1/z$, so we have

$$f(-1/z) = z^k f(z)$$

and

$$f(z+1) = f(z)$$

Remark

One can define modular forms for the subgroups of Γ, for instance, for $\Gamma_0(N)$.
In this case, we add to the definition "level N".
It is clear that a modular form for Γ is of level 1.

Since modular forms are periodic they are perfect number theoretic objects, i.e. f admits a Fourier expansion

Fourier expansion

$$f(z) = \sum_{n \geq 0} a_n q^n$$

where $q = e^{2\pi i z}$, $z \in \mathfrak{U}$.

Properties

Space of modular forms is denoted by $M_k(\Gamma)$, cusp forms by $S_k(\Gamma)$ and they are finite dimensional vector space over $\mathbb{C}$.

Example

The simplest example for the modular forms is *the Ramanujan tau function*:

$$\Delta(z) := \sum_{n\geq 1} \tau(n)q^n = q\prod_{n\geq 1}(1-q^n)^{24} = \eta(z)^{24}$$

which is a modular form of weight 12 for Γ and level 1.

We have

$$\tau(1) = 1, \tau(2) = -24, \tau(3) = 252, \tau(4) = -1472, \tau(5) = 4830, \tau(6) = -6048, ...$$

The Ramanujan-Petersson Conjecture (First version)

Values of tau functions have interesting relations which were observed by Ramanujan, one of them is a trailer for the following minutes of the talk:

Theorem

(Deligne 1974) For all primes p, one has

$$|\tau(p)| \leq 2p^{11/2}$$

Spoiler: Actually, more is true! See upcoming slides..

Eisenstein Series

Another interesting example for modular forms is Eisenstein series.

Definition

An Eisenstein series with half-period ratio z and index k is defined by

$$G_k(z) := \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m+nz)^k}$$

where m and n are not simultaneously equal to 0, $z \in \mathfrak{U}$ and $k > 2$ an integer.

Eisenstein series have the following nice property:

Theorem

For $k \geq 2$, $G_{2k} \in M_{2k}(\Gamma)$.

Hecke Operators

To handle modular forms efficiently, we need *Hecke operators*, which are a certain kind of averaging operators that play an important role in the structure of vector spaces of modular forms and more general automorphic representations.

Definition

For a fixed integer k and any positive integer n, *n_th Hecke operator* is denoted by T_n and it is defined on the set $M_k(\Gamma)$ as $T_n : M_k(\Gamma) \to M_k(\Gamma)$ by

$$(T_n f)(z) := n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{az+bd}{d^2}\right)$$

Note that T_n preserves the space of cusp forms and they have the properties $T_m \circ T_n = T_{mn}$ if $gcd(m,n) = 1$ and for prime p, $T_p \circ T_{p^n} = T_{p^{n+1}} + p^{k-1} T_{p^{n-1}}$.

Hecke Eigenforms

We have the following special modular forms:

Definition

A Hecke eigenform is a modular form which is an eigenvector for all Hecke operators T_m for $m = 1, 2, \ldots$.

Definition

An eigenform is said to be *normalized* when scaled so that the $q-$coefficient in its Fourier series is one:

$$f = a_0 + q + \sum_{n=2}^{\infty} a_n q^n$$

Some Notes

Note that eigenform means simultaneous Hecke eigenform with Γ and existence of eigenforms is a nontrivial result but it comes from the fact that *Hecke algebra* is commutative.
It is clear that coefficients of normalized eigenforms are exactly eigenvalues of Hecke operators, i.e.

$$T_n f = a_n f.$$

Definition

A *newform* is a cuspform of level N that does not come from strictly lower weight.

Definition

L-function of a cusp form f is denoted by $L(s, f)$ and defined as

$$L(s, f) := \sum_{n=1}^{\infty} a_n q^n = \prod_{p|N}(1 - \frac{a_p}{p^s})^{-1} . \prod_{p\nmid N}(1 - \frac{a_p}{p^s} + \frac{1}{p^{2s+1-k}})^{-1}$$

Definition

A newform $f = \sum_{n=1}^{\infty} a_n q^n$ of level N and weight k has complex multiplication if there is a quadratic imaginary field K such that $a_p = 0$ as soon as p is a prime which is inert in K. Otherwise it is defined that f is a newform without complex multiplication.

Preparations for half integral weight modular forms

Let k be an integer. Then $k + 1/2$ is called *half-integer*. We want to define half-integral weight modular forms. We should have similar automorphy factor, namely, $(cz + d)^k$ with the integral weight case which means trouble since we have complex square root function. Hence we should be very careful. Let us start with defining *theta series:*

Definition

For any $z \in \mathfrak{U}$, we define *theta series* by

$$\theta(z) := \sum_{m=-\infty}^{\infty} e^{i\pi m^2 z}$$

Note that we have

$$\theta(z+2) = \theta(z) \text{ and } \theta(-1/z) = \sqrt{-iz}\,\theta(z).$$

After some work and choosing usual branch for the square root function, one can obtain the following fact:

$$\theta(\gamma(z)) = \left(\tfrac{2c}{d}\right)_2 \epsilon_d^{-1}(cz+d)^{1/2}\theta(z)$$

for any $\gamma \in \Gamma$ with $b, c \equiv 0(\mod 2)$, $\epsilon_d = 1$ or i according to $d \equiv 1$ or $3 \mod 4$, respectively, and $\left(\tfrac{2c}{d}\right)_2$ is the Legendre symbol (including the case $|d| = 1$).

Setting $\tilde{\theta}(z) = \theta(2z)$, we have

Theorem

For any $\gamma \in \Gamma_0(4)$*,*

$$\tilde{\theta}(\gamma(z)) = j(\gamma, z).\tilde{\theta}(z)$$

where automorphy factor

$$j(\gamma, z) = \left(\frac{2c}{d}\right)_2 \epsilon_d^{-1}(cz+d)^{1/2}$$

as defined above.

We are ready to define half-integral weight modular forms:

Definition

Any such function that transforms by $j(\gamma, z)^{2k}$ for k fixed half-integer, and all $\gamma \in \Gamma_0(4N)$ and is holomorphic at each cusp, is called a *half-integral weight modular form*.

Shimura Correspondance

We have the following relation between integral weight modular forms and half-integral weight modular forms via *Shimura lift* due to Shimura and Niwa:
Let $k \geq 2$, $4 \mid N$ be integers, χ a Dirichlet character modulo N s.t. $\chi^2 = 1$ and let $f = \sum_{n=1}^{\infty} a(n)q^n \in S_{k+1/2}(N, \chi)$ be a non-zero cuspidal Hecke eigenform (for operators T_{p^2} for primes $p \nmid N$) of weight $k + \frac{1}{2}$ with real coefficients. We denote *Shimura lift of f* with respect to t (for a fixed squarefree t such that $a(t) \neq 0$) by F_t and define by $F_t = \sum_{n=1}^{\infty} A_t(n)q^n \in S_{2k}(N/2, \chi^2)$ which is the Hecke eigenform (for operators T_p for primes $p \nmid N$) of weight $2k$ corresponding to f.

There is the direct relation between the Fourier coefficients of f and those of its lift F_t, namely

$$A_t(n) := \sum_{d|n} \chi_{t,N}(d) d^{k-1} a(\frac{tn^2}{d^2}), \tag{1}$$

where $\chi_{t,N}$ denotes the character $\chi_{t,N}(d) := \chi(d) \left(\frac{(-1)^k N^2 t}{d}\right)$. As we assume f to be a Hecke eigenform for the Hecke operator T_{p^2}, F_t is an eigenform for the Hecke operator T_p, for all primes $p \nmid N$. In fact, in this case $F_t = a(t)F$, where F is a normalised Hecke eigenform independent of t.

Moreover, from the Euler product formula for the Fourier coefficients of half integral weight modular forms, one obtains the multiplicativity relation for $(m, n) = 1$

$$a(tm^2)a(tn^2) = a(t)a(tm^2n^2). \tag{2}$$

Note that the assumption that χ be (at most) quadratic implies that F_t has real coefficients if f does. Furthermore, the coefficients of F_t satisfy the Ramanujan-Petersson bound (Deligne's Theorem) $|\frac{A_t(p)}{a(t)}| \leq 2p^{k-1/2}$ which was introduced before. We can normalise them by letting

$$B_t(p) := \frac{A_t(p)}{2a(t)p^{k-1/2}} \in [-1, 1].$$

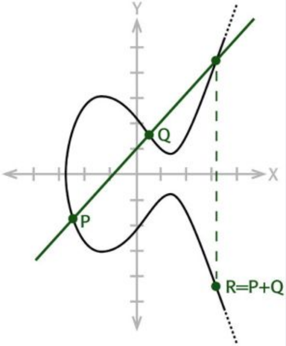
Elliptic Curves

Let $\mathbb{K}$ be a field such that $char(\mathbb{K}) \neq 2, 3$. For fixed $A, B \in \mathbb{Z}$, let us consider the following set:

$$E/K := \{(x, y) \in K : y^2 = x^3 + Ax + B\} \cup \{\infty\}$$

where $\Delta := 4A^3 + 27B^2 \neq 0$.

We can define an addition rule on $E/\mathbb{K}$ for $P, Q \in E/\mathbb{K}$ as



Then with this operation, $E/\mathbb{K}$ forms an abelian group. Note that we need ∞ for technical reasons in group law!

Definition

Let p a prime such that $(p, \Delta) = 1$. Then $E/\mathbb{F}_p$ forms an elliptic curve after reduction. In this case we say *E has a good reduction at p* and p is called *good prime for E*.
Otherwise we say *E has a bad reduction at p* and p is called *bad prime for E*.
Suppose that E has a good reduction at p. Then let us denote the number of points on $E/\mathbb{F}_p$ with N_p, i.e.,

$$N_p := \#\{(x, y) \in \mathbb{F}_p : y^2 = x^3 + Ax + B\} \cup \{\infty\}$$

Definition

$L-$function of an elliptic curve $E/\mathbb{Q}$ is denoted by $L(E,s)$ and defined by

$$L(E,s) := \prod_{p\text{good}} \frac{1}{1-a(p)p^{-s}+p^{1-2s}}$$

where $a(p) := p+1-N_p$.

Modularity Theorem

The following theorem is called *Modularity Theorem* and it is the main bridge between modular forms and elliptic curves which leads to the proof of the Fermat's Last Theorem due to Wiles (1995) with contributions by Taniyama, Shimura, Frey, Serre, Ribet, Taylor and others:

Theorem

Let E be an elliptic curve over $\mathbb{Q}$. Then $L(E,s) = L(s,f)$ for some normalized eigenform of weight 2 for $\Gamma_0(N)$ where N is the conductor of E.

Sato-Tate Conjecture for Elliptic Curves

Theorem

(Harris et al 2010) Let E be an elliptic curve without complex multiplication and p is prime number. Define θ_p by $p+1-\#E(\mathbb{F}_p)=2\sqrt{p}\cos\theta_p$ where $0\leq\theta_p\leq\pi$. Then for $0\leq\alpha<\beta\leq\pi$

$$\lim_{N\to\infty}\frac{\{p : p\leq N, \alpha\leq\theta_p\leq\beta\}}{\{p : p\leq N\}}=\frac{2}{\pi}\int_{\alpha}^{\beta}\sin^2\theta d\theta$$

.

Sato-Tate Conjecture for Modular Forms

Let $f = \sum_{n \geq 1} a(n)q^n$ be a normalised cuspidal Hecke eigenform of weight $2k$ for $\Gamma_0(N)$ without complex multiplication and p a prime number. Then Ramanujan-Petersson bound gives

$$\mid a(p) \mid \leq 2p^{k-1/2}.$$

Using this bound we can make a normalisation on the Fourier coefficients of this Hecke eigenform, namely,

$$b(p) := \frac{a(p)}{2p^{k-1/2}} \in [-1, 1].$$

One defines the *Sato-Tate measure* μ to be the probability measure on the interval $[-1, 1]$ given by $\frac{2}{\pi} \int_a^b \sqrt{1 - t^2} dt$.

Sato-Tate Conjecture for Modular Forms

So we are ready to state the Sato-Tate conjecture for modular forms:

Theorem

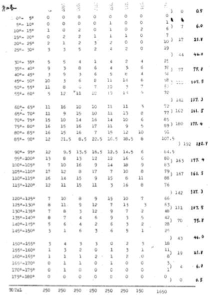
(Barnet-Lamb et al 2011) Let $k \geq 1$ and let $f = \sum_{n\geq 1} a(n)q^n$ be a normalised cuspidal Hecke eigenform of weight $2k$ for $\Gamma_0(N)$ without complex multiplication. Then the numbers $b(p) = \frac{a(p)}{p^{k-1/2}}$ are μ-equidistributed in $[-1, 1]$, when p runs through the primes not dividing N.

Sato-Tate Conjecture for Modular Forms

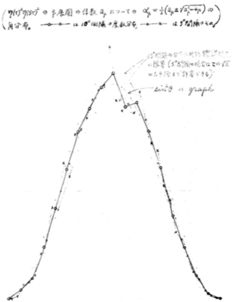
As a corollary of this result, we have the following:
Let $[a, b] \subseteq [-1, 1]$ be a subinterval and $S_{[a,b]} := \{p : (p, N) = 1, b(p) \in [a, b]\}$. Then $S_{[a,b]}$ has natural density equal to $\frac{2}{\pi}\int_a^b \sqrt{1-t^2}dt$.

BEFORE: A guess by Sato

Following graph is originally drawn by Mikio Sato around 1960s by computations on Dedekind-eta products.

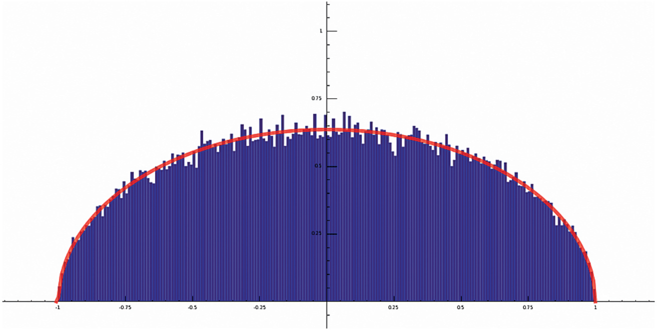


									0	0.5
0°– 5°	0	0	0	0	0	0	0	0		
5°– 10°	0	0	0	0	1	0	0	1		
									7	6.0
10°– 15°	1	0	2	0	1	0	2	6		
15°– 20°	0	2	2	1	1	1	0	7		
									17	21.8
20°– 25°	2	1	2	3	2	0	0	10		
25°– 30°	3	3	5	2	4	2	0	19		
									44	46.0
30°– 35°	5	5	4	1	4	2	4	25		
35°– 40°	9	3	8	6	4	3	6	39		
									77	75.8
40°– 45°	3	9	3	6	5	[illegible]	4	[illegible]		
45°– 50°	10	3	6	8	11	14	6	58		
									[illegible]	107.5
50°– 55°	11	8	6	7	10	3	7	[illegible]		
55°– 60°	5	12	*11	10	10	13	4	70		
									142	137.3
60°– 65°	11	16	10	10	11	11	3	72		
65°– 70°	11	9	15	10	11	13	8	77		
									162	161.5
70°– 75°	15	10	14	16	14	10	6	85		
75°– 80°	16	15	16	17	11	9	5	89		
									180	177.4
80°– 85°	16	15	16	7	15	12	10	91		
85°– 90°	12	21.5	8.5	22.5	16.5	18.5	6	107.5		
									192	182.7
90°– 95°	12	9.5	13.5	16.5	12.5	14.5	6	84.5		
95°–100°	13	8	13	12	12	16	6	80		
									163	177.4
100°–105°	7	10	16	9	14	18	9	83		
105°–110°	17	12	8	17	7	10	8	79		
									167	161.5
110°–115°	16	14	15	9	15	6	11	88		
115°–120°	12	11	15	11	3	16	8	76		
									142	137.3
120°–125°	7	10	8	9	15	10	7	66		
125°–130°	8	11	9	12	7	13	3	63		
									111	107.5
130°–135°	7	8	3	12	9	7	2	48		
135°–140°	8	7	4	6	9	3	5	42		
									70	75.8
140°–145°	5	6	4	2	6	3	2	28		
145°–150°	3	1	6	3	6	5	1	25		
									43	46.0
150°–155°	3	4	3	3	0	2	3	18		
155°–160°	1	3	2	0	1	3	1	11		
									19	21.8
160°–165°	1	1	1	2	1	2	0	8		
165°–170°	0	1	1	0	1	0	0	3		
									4	6.0
170°–175°	0	1	0	0	0	0	0	1		
175°–180°	0	0	0	0	0	0	0	0		
									0	0.5
TOTAL	250	250	250	250	250	250	150	1650		



AFTER

Following graph is drawn by William Stein and it illustrates the Sato-Tate Conjecture for the modular form Δ for $p < 1.000.000$.



On way to the proof

- 1920s, Class Field Theory
- 1933 Hasse's Theorem $\mid p+1-\#E(\mathbb{F}_p)\mid \leq 2\sqrt{p}$
- 1955 Taniyama-Shimura Conjecture
- 1957 Compatible system of l-adic Galois representation
- 1963 Sato's Conjecture, Tate's Conjecture
- 1968 Abel l-adic Representations and Elliptic Curves by Serre. Symmetric power L-functions $L(s, E, \mathrm{Sym}^n)$.

On way to the proof

- 1970s Automorphic Reps and L-function
  Langlands' principle of functionality, Langlands Conjecture
  Weil Conjecture, Ramanujan-Petersson Conjecture
- 1980s Development of Shimura varieties
  Iwasawa main conjecture
  Deformisation of Galois reps and p-adic Hodge Theory
- 1990 Proof of Taniyama-Shimura Conjecture (for semistable curves) which implies FLT.
- 2001 Full proof of Taniyama-Shimura Conjecture

On way to the proof

In April 2006 Clozel, Harris, Shepherd-Barron, Taylor announced a proof of the Sato-Tate Conjecture for elliptic curves

$$E : y^2 = x^3 + ax + b$$

where $a, b \in \mathbb{Z}$ such that $4a^3 + 27b^2 \neq 0$ when $j(E) = \frac{1728 \cdot 4a^3}{4a^3+27b^2}$ is not an integer.

They necessarily proved similar results for elliptic curves over all totally real fields simultaneously whose j-invariants are not algebraic integers.

Achievement

Finally, in July 2009, Barnet-Lamb, Geraghty, Harris, Taylor announced a proof of the Sato-Tate Conjecture for

- all non-CM elliptic curves over total real fields,
- all non-CM modular forms.

Achievement

In October 2009, Barnet-Lamb, Gee, Geraghty obtained a proof of the Sato-Tate Conjecture for all non-CM Hilbert modular forms on $GL(2)$ over totally real fields.

Prize winner

Richard Taylor received "Breakthrough Prizes in Mathematics" for his breakthrough results in theory of automorphic forms who is awarded with 3 million dollars in 2015.

The Bruinier-Kohnen Sign Equidistribution Conjecture

Conjecture

(Bruinier-Kohnen) Assume the set up above. Then

$$\lim_{x\to\infty} \frac{|\{n\leq x : a(n)>0\}|}{|\{n\leq x : a(n)\neq 0\}|} = \frac{1}{2}$$

$$\lim_{x\to\infty} \frac{|\{n\leq x : a(n)<0\}|}{|\{n\leq x : a(n)\neq 0\}|} = \frac{1}{2}$$

Theorem

(I, Wiese 2013), (Arias-de-Reyna, I, Wiese 2015) Assume the set-up above and define the set of primes

$$\mathbb{P}_{>0} := \{p : a(tp^2) > 0\}$$

and similarly $\mathbb{P}_{<0}$ *and* $\mathbb{P}_{=0}$ *(depending on* f *and* t*). Then the sets* $\mathbb{P}_{>0}$ *and* $\mathbb{P}_{<0}$ *have positive natural densities and the set* $\mathbb{P}_{=0}$ *has natural density* 0*.*

The following theorem is a consequence of the Halász' theorem:

Theorem

(I, Wiese 2015) Let $g : \mathbb{N} \to \{-1, 0, 1\}$ *be a multiplicative function. If* $\sum_{p, g(p)=0} \frac{1}{p}$ *converges and* $\sum_{p, g(p)=-1} \frac{1}{p}$ *diverges then*

$$\lim_{x\to\infty} \frac{|\{n \le x : g(n) \gtrless 0\}|}{|\{n \le x : g(n) \neq 0\}|} = \frac{1}{2}.$$

Weakly Regular Set of Primes

Definition

Let S be a set of primes. It is called *weakly regular* if there is $a \in \mathbb{R}$ (called the *Dirichlet density* of S) and a function $g(z)$ which is holomorphic on $\{\mathrm{Re}(z) > 1\}$ and continuous (in particular, finite) on $\{\mathrm{Re}(z) \geq 1\}$ such that

$$\sum_{p \in S} \frac{1}{p^z} = a \log\left(\frac{1}{z-1}\right) + g(z).$$

Proposition

(I, Wiese 2015) Assume the setup above. Then the set $\mathbb{P}_{=0}$ is weakly regular.

Proof.

Combine a result of Serre on Chebotarev sets with some results in (Arias-de-Reyna, I, Wiese 2015). $\square$

We now use the application of Halász' theorem and the weak regularity of $\mathbb{P}_{=0}$ to prove the equidistribution result we are after in terms of natural density.

Theorem

(I, Wiese 2015) Assume the setup above. Then the sets $\{n \in \mathbb{N} | a(tn^2) > 0\}$ and $\{n \in \mathbb{N} | a(tn^2) < 0\}$ have equal positive natural density, that is, both are precisely half of the natural density of the set $\{n \in \mathbb{N} | a(tn^2) \neq 0\}$.

Proof.

Let $g(n) = \begin{cases} 1 & \text{if } a(tn^2) > 0, \\ 0 & \text{if } a(tn^2) = 0, \\ -1 & \text{if } a(tn^2) < 0. \end{cases}$ Due to the relations $a(tn^2m^2)a(t) = a(tn^2)a(tm^2)$ for $gcd(n, m) = 1$, it is clear that $g(n)$ is multiplicative. Since $\mathbb{P}_{=0}$ is weakly regular by the previous Proposition, it follows that $\sum_{p \in \mathbb{P}_{=0}} \frac{1}{p}$ is finite. Moreover, the fact that $\mathbb{P}_{<0}$ is of positive density implies that $\sum_{p \in \mathbb{P}_{<0}} \frac{1}{p}$ diverges. Thus the result follows from the previous Theorem above. □

Open Problems

- Prove the Bruinier-Kohnen Sign Equidistribution Conjecture for all cases.
- Find a Sato-Tate like conjecture for the half integral weight modular eigenforms.

Work in Progress

With W. Kohnen (Heidelberg) and G. Wiese (Luxembourg) and H. Cohen (Bordeaux) (since yesterday ☺)

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Thank you for your attention

Merci á tous!