Using approximate functional equations to build L functions

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Clermont-Ferrand – 20 juin 2017
Example : elliptic curves

Consider an elliptic curve $E/\mathbb{Q}$ of conductor $N$ and root number $\varepsilon = -1$. The associated modular form

$$f = \sum a_n q^n, q = e^{2i\pi z}.$$  \hfill (1)

satisfies $Wf = -f$ and vanishes at $q = e^{-\frac{2\pi}{\sqrt{N}}},$

$$f(q) = 0 = q + a_2 q^2 + a_3 q^3 + \ldots.$$  \hfill (2)
Example: elliptic curves

Consider an elliptic curve $E/\mathbb{Q}$ of conductor $N$ and root number $\varepsilon = -1$. The associated modular form

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By Hasse, $|a_n| \leq \lfloor \sigma_0(n) \sqrt{n} \rfloor \leq n$, so the equality is possible only if

$$q \leq \sum_{k \geq 2} nq^n = \frac{q}{(1 - q)^2} - q = \frac{q^2(2 - q)}{(1 - q)^2}. \quad (3)$$
\[ b(q)=q \cdot (2-q)/(1-q)^2; \]
\[ \text{plot}(N=.5, 40, b(\exp(-2\cdot\pi/\sqrt{N}))) \]
\[ b(q) = q(2-q)/(1-q)^2; \]
\[ \text{plot}(N=.5,40,b(\exp(-2*\pi/\sqrt{N}))) \]
\[ \text{solve}(N=.5,40,b(\exp(-2*\pi/\sqrt{N}))-1) \]
26.181852174699964975652391885916899331

**Theorem**

*If an elliptic curve has rank \( r \geq 1 \), its conductor satisfies \( N \geq 27 \).*
Least conductor: generalize

Degree 2 L-function \( L(s) \), one gamma factor \( \Gamma_C(s) \), conductor \( N \), weight \( k \) and sign \( \varepsilon \). The (symmetrized) inverse Mellin transform

\[
F(x) = e^{\frac{x}{2}} \sum a_n e^{\frac{2\pi}{\sqrt{N}} e^x n}
\]

satisfies

\[
F(x) = \varepsilon F(-x).
\]

In particular for all \( 0 < y < \frac{\pi}{2} \), \( F(iy) - \varepsilon F(-iy) = 0 \).

Let \( t = e^{i\frac{k}{2}} \), \( q = e^{-\frac{2\pi}{\sqrt{N}} e^{iy}} \), one must have

\[
\sum_{n \geq 1} a_n (tq^n - \varepsilon tq^n) = 0
\]
Least conductor : generalize

Using

- $|a_n| \leq \left| \sigma_0(n) n^{\frac{k-1}{2}} \right| \leq \sqrt{3} n^{\frac{k}{2}}$
- $\sum_{n>K} n^{\frac{k}{2}} |q|^n \leq \frac{\sqrt{K+1}^k}{1-\frac{k}{2\alpha y(K+1)}} |q|^K = B_K(q)$ if $2\alpha y(K+1) > k$

one must have ($t = e^{iy\frac{k}{2}}, q = e^{-\frac{2\pi}{\sqrt{N}} e^{iy}}$)

$$|tq - \epsilon \overline{tq}| \leq \sum_{n=2}^{K} \left| \sigma_0(n) n^{\frac{k-1}{2}} \right| |tq^n - \epsilon \overline{tq^n}| + 2\sqrt{3} B_K(q)$$
Odd elliptic curves

\[ y = 0.1 \Rightarrow N \geq 25.63 \text{ better!} \]
Odd elliptic curves

\[ y = 0.2 \Rightarrow N \geq 27.40 \text{. better!} \]
Odd elliptic curves

\[ y = 0.3 \Rightarrow N \geq 29.77 \ldots \text{better!} \]
Odd elliptic curves

\[ y = 0.4 \Rightarrow N \geq 31.58 \ldots \text{better!} \]
Odd elliptic curves

\[ y = 0.5 \Rightarrow N \geq 32.53 \quad \text{.... better!} \]
Odd elliptic curves

\[ y = 0.6 \Rightarrow N \geq 29.57 \ldots \text{too bad!} \]
y = 0.7 \Rightarrow N \geq 26.48 \ldots and worse
Odd elliptic curves

\[ y = 0.8 \Rightarrow N \geq 20.81 \]. and worse
Odd elliptic curves

\[ y = 0.9 \Rightarrow N \geq 17.95 \text{ and worse} \]
• Equation $F(0) = 0 \leadsto N \geq 27$.
• Equation $F(iy) + F(-iy) = 0, y = .5 \leadsto N \geq 33$

**Theorem**

*If an elliptic curve has rank $r \geq 1$, its level satisfies $N \geq 33$. (was $N \geq 27$ previously)*
Automatic results

Plot on $y$ for the least value of $N$ satisfying inequality.

\[ k = 2, \; \varepsilon = -1 \quad N \geq 33 \]
Automatic results

Plot on $y$ for the least value of $N$ satisfying inequality.

$k = 2, \varepsilon = +1 \quad N \geq 11$
Automatic results

Plot on $y$ for the least value of $N$ satisfying inequality.

$k = 4$, $\varepsilon = +1$ \quad N \geq 5
Automatic results

Plot on $y$ for the least value of $N$ satisfying inequality.

$k = 6, \epsilon = +1, N \geq 3$
Automatic results

Plot on \( y \) for the least value of \( N \) satisfying inequality.

\[
k = 8, \ \varepsilon = +1 \quad N \geq 2
\]
Automatic results

Plot on $y$ for the least value of $N$ satisfying inequality.

\[ k = 10, \quad \varepsilon = +1 \quad \ldots \text{hum} \]
Bifurcation

the ratio is not monotonous and crosses the 1-axis several times.

Still the result $N > 1.7$ is correct, combining several plots.
Results for degree 2

weight $k$ and level $N$, root number $\epsilon$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\epsilon$</th>
<th>bound</th>
<th>LMFDB object</th>
<th>accurate</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>$N &gt; 10.45$</td>
<td>11a</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>$N &gt; 32.95$</td>
<td>37a</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>$N &gt; 4.63$</td>
<td>5.4.1a</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>-1</td>
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<td>13.4.1a</td>
<td>✓</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>$N &gt; 2.85$</td>
<td>3.6.1a</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>$N &gt; 6.55$</td>
<td>7.6.1b</td>
<td>✓</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>$N &gt; 1.95$</td>
<td>2.8.1a</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>$N &gt; 4.09$</td>
<td>5.8.1a</td>
<td>✓</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>$N &gt; 1.37$</td>
<td>2.10.1a</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>$N &gt; 2.75$</td>
<td>3.10.1b</td>
<td>✓</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>$\star$</td>
<td>$\Delta$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>$N &gt; 1.97$</td>
<td>4.12.1a</td>
<td></td>
</tr>
</tbody>
</table>
$k=12$ : Ramanujan $\Delta$ function

$x = 0.05i$, $N \geq 0.86$
$k=12$ : Ramanujan $\Delta$ function

$x = 0.20i, N \geq 0.88$
$k=12 : \text{Ramanujan } \Delta \text{ function}$

$x = 0.25i, N \geq 0.90$
k=12 : Ramanujan Δ function

$x = 0.35i, N \geq 0.93$
k=12 : Ramanujan $\Delta$ function

$x = 0.40i, N \geq 0.94$
$k=12$ : Ramanujan $\Delta$ function

$x = 0.45i, \ N \geq 0.96$
k = 12 : Ramanujan $\Delta$ function

$3.9056 - 9.1369 0.4 3$

$x = 0.50i, N \geq 0.99$
$k=12 : \text{Ramanujan } \Delta \text{ function}$

$x = 0.55i, \quad 0.90 \leq N \leq 1.01 \text{ or } N \geq 1.18$
\[ k=12 : \text{Ramanujan} \ \Delta \ \text{function} \]

\[ x = 0.60i, \ 0.84 \leq N \leq 1.04 \ \text{or} \ N \geq 1.27 \]
$k=12 : \text{Ramanujan } \Delta \text{ function}$

$$x = 0.70i, \ 0.78 \leq N \leq 1.12 \text{ or } N \geq 1.37$$
\( k=12 : \text{Ramanujan } \Delta \text{ function} \)

\[ x = 0.80i, N \geq 0.75 \]
$k=12 : \text{Ramanujan } \Delta \text{ function}$

$x = 0.90i, N \geq 0.74$
k=12 : Ramanujan $\Delta$ function

$x = 1.00i, N \geq 0.55$
From an analytic point of vue, it’s a miracle that $\Delta$ exists. It could not for $N < .9999$, nor $1.00001 < N < 1.37$. 
General L functions

- Dirichlet series $L(s) = \sum_{n \geq 1} a_n n^{-s}$
- Gamma factor of level $N$ and degree $d$

$$\gamma(s) = N^s \prod_{j=1}^d \Gamma_R(s + \lambda_i)$$

- Functional equation of weight $k$

$$\Lambda(s) = L(s) \gamma(s) = \epsilon \overline{\Lambda}(k - s)$$

- Ramanujan bound $a_n \leq \sigma_0(n)^{d-1} n^{\frac{k-1}{2}}$
- + $\Lambda$ meromorphic. Here assume holomorphic.
Theta equations

Fourier form

\[ \Lambda(s) = \epsilon \Lambda(k - s) \]

if and only if for all \( x \in \mathbb{R}^+ \) \( - \frac{d\pi}{4}, \frac{d\pi}{4} \],

\[ F(x) = \epsilon F(-x) \]

where

\[ F(x) = e^{\frac{kx}{2}} \sum_n a_n M^{-1} [\gamma(s); e^{x} n] \] (6)

\[ = \frac{1}{2\pi} \int_{\mathbb{R}} \Lambda(\frac{k}{2} + it)e^{ixt} \, dt \] (7)
Inverse Mellin transforms

| $\gamma(s)$ | $N^s_2 \Gamma_{\mathbb{R}}(s)$ | $N^s_2 \Gamma_{\mathbb{C}}(s)$ | $N^s_2 \Gamma_{\mathbb{C}}(s) \Gamma_{\mathbb{C}}(s + \nu)$ |
| $\mathcal{M}^{-1} [\gamma; x]$ | $e^{-\frac{\pi}{N} x^2}$ | $e^{-\frac{2\pi}{\sqrt{N}} x}$ | Bessel $K_\nu$ |

More gamma factors (real shifts) now available in Pari/gp

```python
# More gamma factors available in Pari/gp

# New function to compute inverse Mellin transform

g = gammamellininvinit([0, 0, 1, 1, 1])
gammamellininv(g, x)
```

**Conclusion** For any L function, easy to produce equations

$$\sum_n a_n x_n$$ satisfied by the Dirichlet series, with $x_n \to 0$ exponentially.
Theorem (Smallest discriminants of number fields)

Let $K/Q$ be a number field of signature $r$, $s$.

- if $r, s = 3, 0$, then $|\Delta| \geq 25$;
- if $r, s = 1, 1$, then $|\Delta| \geq 15$;
- if $r, s = 4, 0$, then $|\Delta| \geq 105$;
- if $r, s = 2, 1$, then $|\Delta| \geq 64$;
- if $r, s = 0, 2$, then $|\Delta| \geq 40$;
- if $r, s = 5, 0$, then $|\Delta| \geq 356$. 
"Proof" (case 4,0)

\[ \zeta^*_K(s) = \frac{\zeta_K(s)}{\zeta(s)} \] is degree 4, weight 1, holomorphic \( L \) function, with \( \gamma(s) = N^s \Gamma_R(s)^4 \).
**Example: weight \( k=1 \)**

**First weight 1 L functions**

<table>
<thead>
<tr>
<th>gamma</th>
<th>N</th>
<th>Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0]</td>
<td>5, 8, 12, ...</td>
<td>((\frac{5}{3}), (\frac{8}{3}), (\frac{12}{3}), ...)</td>
</tr>
<tr>
<td>[1]</td>
<td>3, 4, 7, ...</td>
<td>((-\frac{3}{3}), (-\frac{4}{3}), (-\frac{7}{3}), ...)</td>
</tr>
<tr>
<td>[0, 0]</td>
<td>25, 40, 49, ...</td>
<td>((\frac{5^2}{3}), (\frac{5\times8}{3}), \zeta_{x^3-x^2-2x+1}^*, \ldots)</td>
</tr>
<tr>
<td>[0, 1]</td>
<td>15, 20, 23, ...</td>
<td>((-\frac{3\times5}{3}), (\frac{-4\times5}{3}), \zeta_{x^3-x^2+1}^*, \ldots)</td>
</tr>
<tr>
<td>[1, 1]</td>
<td>9, 12, 16, ...</td>
<td>((-\frac{3^2}{3}), (\frac{-3\times-4}{3}), (\frac{-4\times-4}{3}), \ldots)</td>
</tr>
<tr>
<td>[0, 0, 0]</td>
<td>125, 200, 245, ...</td>
<td>((\frac{5^3}{3}), (\frac{5^2\times8}{3}), (\frac{5}{3}), \zeta_{x^3-x^2-2x+1}^*, \ldots)</td>
</tr>
<tr>
<td>[0, 0, 1]</td>
<td>75, 100, 115, ...</td>
<td>((-\frac{3\times5^2}{3}), (\frac{-4\times5^2}{3}), (\frac{5}{3}), \zeta_{x^3-x^2+1}^*, \ldots)</td>
</tr>
<tr>
<td>[0, 1, 1]</td>
<td>45, 60, 69, ...</td>
<td>((-\frac{3^2\times5}{3}), (\frac{-3\times-4\times5}{3}), (\frac{-3}{3}), \zeta_{x^3-x^2+1}^*, \ldots)</td>
</tr>
<tr>
<td>[1, 1, 1]</td>
<td>27, 36, 48, ...</td>
<td>((-\frac{3^3}{3}), (\frac{-3^2\times-4}{3}), (\frac{-3\times(-4)^2}{3}), \ldots)</td>
</tr>
</tbody>
</table>
Enumerate Dirichlet series

Recall: from the beginning we use

\[ |q| \leq \sum_{k \geq 2} b_k |q^k|, \text{ where } |a_k| \leq b_k. \quad (8) \]

Could push the game further trying values of \( a_2 \) in the left-hand side...

\[ |q + a_2 q^2| \leq \sum_{k \geq 3} b_k |q^k|, \text{ where } |a_k| \leq b_k. \quad (9) \]

for all values \( a_2 \in [-b_2, b_2] \).
Enumerate Dirichlet series

- **Goal**: find all L-functions having specified invariants
- **Input**: an approximate functional equation \( \sum_n a_n x_n = 0 \)
- **Hypothesis**: \( a_n \in \mathbb{Z} + \) Ramanujan bounds \( |a_n| \leq b_n \).

1. \( a_1 = 1 \)
2. \( a_2 \in [-b_2, b_2] \) s.t.
   \[ |(a_1 x_1) + a_2 x_2| \leq B_2 = b_3 |x_3| + b_4 |x_4| + b_5 |x_5| + b_6 |x_6| + b_7 |x_7| + \]
3. \( a_3 \in [-b_3, b_3] \) s.t.
   \[ |(a_1 x_1 + a_2 x_2) + a_3 x_3| \leq B_3 = b_4 |x_4| + b_5 |x_5| + b_6 |x_6| + b_7 |x_7| + \]
4. \( \ldots \)
Enumerate Dirichlet series

goal find all L-functions having specified invariants

input an approximate functional equation $\sum_n a_n x_n = 0$

hypothesis $a_n \in \mathbb{Z} +$ Ramanujan bounds $|a_n| \leq b_n.$

1. $a_1 = 1$
2. $a_2 \in [-b_2, b_2]$ s.t.
   \[
   |(a_1 x_1) + a_2 x_2| \leq B_2 = b_3 |x_3| + b_4 |x_4| + b_5 |x_5| + b_6 |x_6| + b_7 |x_7| + \ldots
   \]
3. $a_3 \in [-b_3, b_3]$ s.t.
   \[
   |(a_1 x_1 + a_2 x_2) + a_3 (x_3 + a_2 x_6)| \leq B_3 = b_4 |x_4| + b_5 |x_5| + \ldots + b_7 |x_7| + \ldots
   \]
4. \ldots
More structure on Dirichlet series

- Euler product $L(s) = \prod F_p (p^{-s})^{-1}$
- Euler factor $F_p(T) = 1 + c_{p,1} T + \ldots T^d$
- local functional equation

$$c_{p,d-j} = \chi(p) p^{\frac{w}{2} (d-2j)} c_{p,j}$$

with $\chi$ Dirichlet character modulo $N$

- Ramanujan bounds: using $|\text{roots}| \leq p^{\frac{w}{2}}$

$$c_{p,j} \leq \binom{d}{j} p^{j \frac{w}{2}}$$

$$a_k \leq k^{\frac{w}{2}} \prod_{p^e \| k} \binom{e + d - 1}{d - 1}$$
Search

Idea

• Explore a tree of Dirichlet series (or local Euler factors)
• Search on Euler coefficients $c_{p,e}$ for $e \leq \frac{d}{2}$ (or $e \leq d - 1$ if $p \mid N$).
• Depth-first search (constant in memory)

Examples

gp > lfunbuild([[],1,[0,1],2,11,1],45,[2])
time = 11 ms.
%1 = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
gp > lfunbuild([[],1,[0,1],2,12,1],45,[2])
time = 11 ms.
%2 = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
Idea

- Explore a tree of Dirichlet series (or local Euler factors)
- Search on Euler coefficients $c_{p,e}$ for $e \leq \frac{d}{2}$ (or $e \leq d - 1$ if $p \mid N$).
- Depth-first search (constant in memory)

Examples

```
gp> lfunbuild([[],1,[0,1],2,26,1],59,[1])
time = 7 ms.
%3 = [1, 3, 3, 3, 2, 26, 4, 15, 5, 2, 2, 2, 2, 12, 2, 2, 2]
```
Search

Idea

• Explore a tree of Dirichlet series (or local Euler factors)
• Search on Euler coefficients $c_{p,e}$ for $e \leq \frac{d}{2}$ (or $e \leq d - 1$ if $p \mid N$).
• Depth-first search (constant in memory)

Examples

gp> lfunbuild([[],1,[0,1],2,26,1],59,[1])
time = 7 ms.
%3 = [1, 3, 3, 3, 2, 26, 4, 15, 5, 2, 2, 2, 2, 12, 2, 2, 2]

the choice of equation is still important!

gp> lfunbuild([[],1,[0,1],2,26,1],59,[[.42]])
time = 7 ms.
%4 = [1, 3, 3, 2, 3, 2, 4, 2, 3, 2, 2, 2, 2, 2, 4, 2, 2]
Program lfunbuild

Two independant functions

• prepare search tree
  • compute a family of equations
  • identify search variables, ranges, search levels in the tree
  • craft nice equation for each level
  • compute tails $B_p$

• prune tree using depth first search (constant memory)
  • start at $p = 2$
  • solve $|\text{polynomial}(a_p)| \leq B_p$
  • for each possible value $a_p$
    • propagate value in Dirichlet series
    • recursively descend next level

gp > for(N=10,100,print(N," : ",Vec(lfunbuild([],1,[0,1],2,N,1],31,[2]))))
Write $k \prec p^e$ if $k$ is $p$-smooth and $p^e \nmid k$.
Assume $\{a_k\}$ known for $k \prec p^e$. Equation for coefficient $a_{p^e}$:

$$\left( \sum_{k \prec p^e} a_k x_k \right) + a_{p^e} \left( \sum_{m \prec p} a_m x_{p^e m} \right) = \sum_{k \succ p^e} a_k x_k$$

Ramanujan bound $|a_n| \leq b_n$ on the tail
\[ \rightsquigarrow \text{polynomial equation } |P(a_{p^e})| \leq r \]
\[ \rightsquigarrow \text{solve in integers in } [-b_{p^e}, b_{p^e}]. \]
Write \( k \prec p^e \) if \( k \) is \( p \)-smooth and \( p^e \mid k \).
Assume \( \{a_k\} \) known for \( k \prec p^e \). Equation for coefficient \( a_{p^e} \):

\[
\left( \sum_{k \prec p^e} a_k x_k \right) + a_{p^e} \left( \sum_{m \prec p} a_m x_{p^em} \right) = \sum_{k > p^e} a_k x_k
\]

If \( e \geq \frac{d}{2} \), by reciprocity of \( F_p(T) \) all \( a_{p^\ell} \) in terms of \( a_{p^j}, j \leq e \)

\[
\left( \sum_{n \prec p^e} a_n x_n \right) + \sum_{\ell \geq e} a_{p^\ell} \left( \sum_{m \prec p} a_m x_{p^\ell m} \right) = \sum_{n > p^\infty} a_n x_n
\]

Ramanujan bound \( |a_n| \leq b_n \) on the tail
\( \leadsto \) polynomial equation \( |P(a_{p^e})| \leq r \)
\( \leadsto \) solve in integers in \( [-b_{p^e}, b_{p^e}] \).
Case of polynomial equation of degree 1:

\[ |S_0 + a_p S_1| \leq B_p = b_{p+} |x_{p+}| + b_{p++} |x_{p++}| + \ldots \]

- \( |S_1| \geq B_p \Rightarrow \) at most one solution \( a_p \).
- the ratio \( \frac{B_p}{x_p} \) can be studied a priori
- usually nice at big prime gaps
- can be horrible for twin primes
Combine equations

Combine approximate functional equations:

\[ X_n = (x_{n,1}, \ldots x_{n,r}) \]

Define

\[ S = \sum_{n \prec p} a_n X_n, \quad T = \sum_{m \prec p} a_m X_{p^e m}, \quad \text{and} \quad R = (b_n X_n)_{n \succ p}. \]

For all \( \lambda \in \mathbb{R}^n \),

\[ |S.\lambda + a_p T.\lambda| \leq \|R.\lambda\|_1 \]

Choose \( \lambda \) to minimize \( \frac{\|R.\lambda\|}{\|W.\lambda\|} \): "least absolute deviations". Can be solved with iterated + weighted least squares.
• Ramanujan $\Delta$ very easy to build (despite $k = 12$).

```gp
lfunbuild([[],1,[0,1],12,1,1],100,2)
time = 42 ms.
%1 = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
```

• Same for higher weight, conductor 1
• May need many equations to cancel tail (here 11 $\otimes 2$
• For $N = 66$, $k = 2$ and central character $\chi = ( -66 \cdot \cdot )$, a match exists for the 71 first primes, then disappears.
• Modular forms expansions 805b and 805c start to differ at primes 11 and 13, with values exchanged.
Observations

- Ramanujan $\Delta$ very easy to build (despite $k = 12$).
- Same for higher weight, conductor 1

```
gp > lfunbuild([[]],1,[0,1],20,1,1),20,[2],1)
time = 20 ms.
%4 = [[1, 456, 50652, -316352, -2377410, 23097312, -16917544, -383331840, 1403363637, -1084098960, -16212108, -16023861504, 50421615062, -7714400064, -120420571320, -8939761664, 225070099506, 639933818472, 0, 752098408320]]
```
Observations

- Ramanujan $\Delta$ very easy to build (despite $k = 12$).
- Same for higher weight, conductor 1

```gp
> lfunbuild([], 1, [0, 1], 20, 1, 1, 20, [2], 1)
time = 20 ms.
%4 = [[1, 456, 50652, -316352, -2377410, 23097312, -16917544, -383331840, 1403363637, -1084098960, -16212108, -16023861504, 50421615062, -7714400064, -120420571320, -8939761664, 225070099506, 639933818472, 0, 752098408320]]
```

```gp
> mfcoefs(mfsearch([1, 20])[1][2], 30)
%5 = [0, 1, 456, 50652, -316352, -2377410, 23097312, -16917544, -383331840, 1403363637, -1084098960, -16212108, -16023861504, 50421615062, -7714400064, -120420571320, -8939761664, 225070099506, 639933818472, 0, 752098408320]
```

- May need many equations to cancel tail (here 11 $\otimes 2$)

- For $N = 66$, $k = 2$ and central character $\chi = (\frac{-66}{\cdot})$, a match exists for the 71 first primes, then disappears.
Observations

- Ramanujan $\Delta$ very easy to build (despite $k = 12$).

- May need many equations to cancel tail (here $11a \otimes 2$)

- For $N = 66$, $k = 2$ and central character $\chi = \left(\frac{-66}{.}\right)$, a match exists for the 71 first primes, then disappears. $\left(\frac{-66}{.}\right)$ is trivial up to $p = 19$, and $\pi(19^2) = 72$
Observations

- Ramanujan $\Delta$ very easy to build (despite $k = 12$).

- May need many equations to cancel tail (here $11a^\otimes 2$)

- For $N = 66$, $k = 2$ and central character $\chi = (\frac{-66}{\cdot})$, a match exists for the 71 first primes, then disappears.

- Modular forms expansions 805b and 805c start to differ at primes 11 and 13, with values exchanged.
Conclusion

ToDo
• work still in progress... [bugs, precision]
• optimize equations
• try other sources of equations
• write tree search in ball arithmetic (Arb)

Goals
• compute interesting examples.
  Challenges : $\Gamma_C(s - 6)\Gamma_C(s)$, $k = 20$, $N = 1$, $a_2 = 0$...
• prove nothing exists outside what is expected
• fill missing bad Euler factors rigorously (e.g. symmetric powers)