A Brumer-Stark conjecture for Galois extensions

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X.-F. Roblot
Institut Camille Jordan
Université Claude Bernard – Lyon 1

Joint work with G. Dejou
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The (abelian) Brumer-Stark conjecture: the objects

\[ K/k \text{ abelian extension of number fields} \]

\[ G = \text{Gal}(K/k), \text{its Galois group} \]

\[ S \subset \text{Pl}(k) \text{ finite, containing Pl}_\infty(k) \text{ and Pl}_{\text{ram}}(K/k) \]

For \( \chi \in \hat{G} \), \( L_{K/k,S}(s, \chi) = \prod_{p \not\in S} (1 - \chi(\sigma_p)\mathcal{N}p^{-s})^{-1} \) (Hecke \( L \)-function)

The Brumer-Stickelberger element

\[ \theta_{K/k,S} := \sum_{\chi \in \hat{G}} L_{K/k,S}(0, \bar{\chi})e_\chi \]
The Brumer-Stickelberger element is the only element of $\mathbb{C}[G]$ such that

$$
\chi(\theta_{K/k,S}) = L_{K/k,S}(0, \bar{\chi}), \ \forall \chi \in \hat{G}
$$

Let $v \in S$ and let $N_v = \sum_{\sigma \in D_v} \sigma$. Then

$$
N_v \cdot \theta_{K/k,S} = 0
$$

In particular, if there exists $v \in S$ totally split, then $\theta_{K/k,S} = 0$.

Let $p$ be a prime ideal of $k$ not in $S$. Then

$$
\theta_{K/k,S \cup \{p\}} = \theta_{K/k,S} \cdot (1 - \sigma_p^{-1})
$$

For all $\xi \in \text{Ann}_{\mathbb{Z}[G]}(\mu_K)$, we have $\xi \cdot \theta_{K/k,S} \in \mathbb{Z}[G]$
The Brumer-Stark Conjecture \( \text{BS}(K/k, S) \)

Let \( w_K := |\mu_K| \). We have

\[
w_K \theta_{K/k,S} \in \text{Ann}_{\mathbb{Z}[G]}(\text{Cl}_K)
\]

Furthermore, for any fractional ideal \( \mathfrak{A} \) of \( K \), there exists \( \alpha \in K^\times \) with

\[
\begin{align*}
\mathfrak{A}^{w_K \theta_{K/k,S}} &= (\alpha) \\
\alpha &\in K^\circ \\
K(\alpha^{1/w_K})/k &\text{ is abelian}
\end{align*}
\]

\( \text{BS}(K/k, S; \mathfrak{A}) \)

Remarks.

- \( K^\circ := \{ x \in K^\times : |x|_w = 1, \forall w \in \text{Pl}_\infty(k) \} \) (anti-units)

- If \( K \) not totally complex or \( k \) not totally real, then \( \theta_{K/k,S} = 0 \).
For $\mathcal{A}$ be a fractional ideal of $K$, $\text{BS}(K/k, S; \mathcal{A})$ is equivalent to

1. There exists an extension $L/K$ such that $L/k$ is abelian and an anti-unit $\gamma \in L^\circ$ such that $(\mathcal{A}\mathcal{O}_L)^{\theta_{K/k,S}} = \gamma \mathcal{O}_L$.

2. For almost all prime ideals $p$ of $k$, there exists $\alpha_p \in K^\circ$ such that $\mathcal{A}(\sigma_p - N(p))^{\theta_{K/k,S}} = \alpha_p \mathcal{O}_K$ and $\alpha_p \equiv 1 \pmod{^* p\mathcal{O}_K}$.

3. There exist a family $(a_i)_{i \in I}$ of element of $\mathbb{Z}[G]$ generating $\text{Ann}_{\mathbb{Z}[G]}(\mu_K)$ and a family $(\alpha_i)_{i \in I} \subset K^\circ$ such that $\mathcal{A}^{a_i \theta_{K/k,S}} = \alpha_i \mathcal{O}_K$ and $\alpha_i^{a_j} = \alpha_j^{a_i}$ for all $i, j \in I$.

The set $\{\mathcal{A} : \text{BS}(K/k, S; \mathcal{A}) \text{ is true}\}$ is a group stable under the action of $G$ that contains the principal ideals of $K$.

Assume $k \subset F \subset K$. Then $\text{BS}(K/k, S) \implies \text{BS}(F/k, S)$.
A Brumer-Stark conjecture for Galois extensions

The (abelian) Brumer-Stark conjecture: proved cases

\[ \text{BS}(K/k, S) \] is true in the following cases

- \( k = \mathbb{Q} \). [Stickelberger theorem]
- \( K \) is principal. [Tate]
- \( K/k \) of degree 4 contained in \( K/k_0 \) Galois but not abelian of degree 8. [Tate]
- \( G \) is of exponent 2 (+ some technical conditions). [Sands]
- Many numerical cases with \( k \) of degree 2, 3 or 4. [Greither, R., Tangedal]

Base change [Hayes]. Let \( K/k'/k \) abelian.

Then \( \text{BS}(K/k, S) \implies \text{BS}(K/k', S') \) with \( S' := \{ v' \in \text{Pl}(k') : v'_{|k} \in S \} \).
The local Brumer-Stark Conjecture $BS^{(\ell)}(K/k, S)$

Let $w_{K,\ell}$ be the $\ell$-part of $w_K$. We have

$$w_K \theta_{K/k, S} \in \text{Ann}_{\mathbb{Z}[G]}(\text{Cl}_K \{\ell\})$$

Furthermore, for $\mathcal{A}$ with $[\mathcal{A}] \in \text{Cl}_K \{\ell\}$, there exists $\alpha \in K^\times$ with

- $\mathcal{A}^{w_K \theta_{K/k, S}} = (\alpha)$
- $\alpha \in K^\circ$
- $K(\alpha^{1/w_{K,\ell}})/k$ is abelian

We have

$$BS(K/k, S) \iff BS^{(\ell)}(K/k, S) \quad \forall \ell$$

Proved in many cases of degree $2p$ [Greither, R., Tangedal; Smith].

$BS^{(\ell)}(K/k, S)$ is proved for $\ell \neq 2$ by Popescu-Greither provided the adequate Iwasawa $\mu$-invariant vanishes.
In order to generalize the Brumer-Stark conjecture to the Galois case, we need to generalize the following:

The **Brumer-Stickelberger element**: \( \theta_{K/k,S} := \sum_{\chi \in \hat{G}} L_{K/k,S}(0, \bar{\chi}) e_{\chi} \)

The **Brumer part**: \( w_K \theta_{K/k,S} \in \text{Ann}_{\mathbb{Z}[G]}(\text{Cl}_K) \)

The **Stark part**: \( \mathcal{A}^{w_K \theta_{K/k,S}} = (\alpha) \) with \( \alpha \in K^\circ \) and \( K(\alpha^{1/w_K})/k \) abelian

(Another generalization in a different direction has been done by A. Nickel)

**We assume from now on that** \( K/k \) **is a Galois extension with group** \( G \).
After Hayes, define

$$
\theta_{K/k,S} := \sum_{\chi \in \hat{G}} L_{K/k,S}(0, \bar{\chi}) e_{\chi} \in Z(\mathbb{C}[G])
$$

where $\hat{G}$ is the set of irreducible characters of $G$ and

$$
e_{\chi} := \frac{\chi(1)}{|G|} \sum_{g \in G} \bar{\chi}(g) g
$$

are the idempotents of $Z(\mathbb{C}[G])$.

Clearly, we recover the previous definition of $\theta_{K/k,S}$ when $K/k$ is abelian.
Recall the properties of the Brumer-Stark element in the abelian case.

The Brumer-Stickelberger element is the only element of $\mathbb{C}[G]$ such that

$$\chi(\theta_{K/k,S}) = L_{K/k,S}(0, \bar{\chi}), \quad \forall \chi \in \hat{G}$$

Let $v \in S$ and let $N_v = \sum_{\sigma \in D_v} \sigma$. Then

$$N_v \cdot \theta_{K/k,S} = 0$$

In particular, if there exists $v \in S$ totally split, then $\theta_{K/k,S} = 0$.

Let $p$ be a prime ideal of $k$ not in $S$. Then

$$\theta_{K/k,S \cup \{p\}} = \theta_{K/k,S} \cdot (1 - \sigma_p^{-1})$$

For all $\xi \in \text{Ann}_{\mathbb{Z}[G]}(\mu_K)$, we have $\xi \cdot \theta_{K/k,S} \in \mathbb{Z}[G]$.
Let $\mathcal{C}$ be the set of conjugacy classes of $G$.

Recall that $Z(\mathbb{C}[G]) = \mathbb{C}[C : C \in \mathcal{C}]$.

For $\chi \in \hat{G}$, the map $\Phi_{\chi} : Z(\mathbb{C}[G]) \to \mathbb{C}$ defined by $\Phi_{\chi}(C) = \frac{\chi(C)}{\chi(1)}$ is a (ring) homomorphism from $Z(\mathbb{C}[G])$ to $\mathbb{C}$.

The Brumer-Stickelberger element is the only element of $Z(\mathbb{C}[G])$ such that

$$\Phi_{\chi}(\theta_{K/k,S}) = L_{K/k,S}(0, \overline{\chi}), \ \forall \chi \in \hat{G}$$
Let \( v \in S \) and let \( N_v = \sum_{\sigma \in D_w} \frac{1}{|C_\sigma|} \in Z(C[G]) \) where \( w \mid v \) and \( C_\sigma \) is the conjugacy class of \( \sigma \). Then

\[
N_v \cdot \theta_{K/k,S} = 0
\]

In particular, if there exists \( v \in S \) totally split, then \( \theta_{K/k,S} = 0 \).

Furthermore, for all complex conjugation \( \tau \) in \( G \), we also have

\[
(1 + \tau) \cdot \theta_{K/k,S} = 0
\]

Let \( p \) be a prime ideal of \( k \) not in \( S \). Then

\[
\theta_{K/k,S \cup \{p\}} = \theta_{K/k,S} \cdot \sum_{\chi \in \hat{G}} \det(1 - \rho_\chi(\sigma_p)) e_\chi
\]
It follows from the \textbf{principal rank zero Stark conjecture} (proved by Tate) that

\[ \theta_{K/k,S} \in \mathbb{Q}[G] \]

Explicit examples show that \( w_K \theta_{K/k,S} \not\in \mathbb{Z}[G] \) in general...

We make the following \textbf{assumption}:

Let \( m_G := \operatorname{lcm}_{C \in \mathcal{G}} |C| \), then

\[ m_G \xi \cdot \theta_{K/k,S} \in \mathbb{Z}[G], \ \forall \xi \in \operatorname{Ann}_{\mathbb{Z}[G]}(\mu_K) \]

Note that \( m_G = 1 \) if and only if \( G \) is abelian.
The Galois Brumer-Stark Conjecture $\text{BS}_{\text{Gal}}(K/k, S)$

We have

$$m_G w_K \theta_{K/k, S} \in \text{Ann}_{\mathbb{Z}[G]}(\text{Cl}_K)$$

Furthermore, for any fractional ideal $\mathfrak{A}$ of $K$, there exists $\alpha \in K^\times$ with

- $\mathfrak{A}^{m_G w_K \theta_{K/k, S}} = (\alpha)$
- $\alpha \in K^\circ$
- ...

What about the "abelian condition"?
(Recall that it is conjectured that $K(\alpha^{1/w_K})/k$ is abelian when $K/k$ is abelian.)
Consider $G$ as a finite group. A group extension

$$1 \longrightarrow \Delta \longrightarrow \Gamma \xrightarrow{s} G \longrightarrow 1.$$ 

is central if $\Delta < Z(\Gamma)$.

Let

$$[\Gamma, \Gamma] := \langle \gamma_0 \gamma_1 \gamma_0^{-1} \gamma_1^{-1} : \gamma_0, \gamma_1 \in \Gamma \rangle =: [\gamma_0, \gamma_1]$$

be the commutator subgroup of $\Gamma$.

We say the above extension is strong central if $\Delta \cap [\Gamma, \Gamma] = 1$. 

Let $\Gamma$ be a group extension of $G$ by $\Delta$, that is

$$
1 \longrightarrow \Delta \longrightarrow \Gamma \overset{s}{\longrightarrow} G \longrightarrow 1.
$$

We have

- **If the extension is strong central then it is central.**
  
  Proof. Let $\delta \in \Delta$ and $\gamma \in \Gamma$, $s([\gamma, \delta]) = 1$ thus $[\gamma, \delta] \in \Delta$ and $[\gamma, \delta] = 1$.

- **The extension is strong central iff, for any $H < G$ with $H$ abelian, $s^{-1}(H)$ is abelian.**
  
  Proof. Assume strong central. For $\gamma, \gamma' \in s^{-1}(H)$, $s([\gamma, \gamma']) = 1$ so $[\gamma, \gamma'] = 1$ and $s^{-1}(H)$ is abelian. (Other direction: exercise!)

- **If the extension is strong central then $m_{\Gamma} = m_{G}$.**

In particular, for $\Gamma$ a strong central extension of $G$ by $\Delta$, the group $\Gamma$ is abelian if and only if $G$ is abelian.
Go back to our situation: $K/k$ is a Galois extension with group $G$.

An extension $L$ of $K$ is a strong central extension of $K/k$ if $L/k$ is Galois and $\Gamma := \text{Gal}(L/k)$ is a strong central extension of $G$ by $\Delta := \text{Gal}(L/K)$.

Let $L^{ab}$ be the maximal sub-extension of $L/k$ that is abelian over $k$. Then $L$ is a strong central extension of $K/k$ if and only if $L/k$ is Galois and $L = KL^{ab}$.

Furthermore, if $L$ is strong central extension of $K/k$ then $\text{Gal}(L/K) \cong \text{Gal}(L^{ab}/K^{ab})$. 
The Galois Brumer-Stark Conjecture $\mathbf{BS}_{\text{Gal}}(K/k, S)$

We have

$$m_{G,w_K} \theta_{K/k,S} \in \text{Ann}_{\mathbb{Z}[G]}(\text{Cl}_K)$$

Furthermore, for any fractional ideal $\mathfrak{A}$ of $K$, there exists $\alpha \in K^\times$ with

- $\mathfrak{A}^{m_{G,w_K} \theta_{K/k,S}} = (\alpha)$
- $\alpha \in K^\circ$
- $K(\alpha^{1/w_K})$ is a strong central extension of $K/k$

$$\mathbf{BS}_{\text{Gal}}(K/k, S; \mathfrak{A})$$

When $K/k$ is abelian, we have $\mathbf{BS}(K/k, S) \iff \mathbf{BS}_{\text{Gal}}(K/k, S)$. 
A Brumer-Stark conjecture for Galois extensions

The Galois Brumer-Stark conjecture: generalization of Tate’s theorem

For $\mathfrak{A}$ be a fractional ideal of $K$, $\text{BS}_{\text{Gal}}(K/k, S; \mathfrak{A})$ is equivalent to

1. There exists an extension $L/K$ such that $L$ is a strong central extension of $K/k$ and $\gamma \in L^\circ$ with $(\mathfrak{A}\mathcal{O}_L)^{m_G \theta_{K/k,S}} = \gamma \mathcal{O}_L$.

2. For almost all prime ideals $\mathfrak{p}$ of $K$, there exists $\alpha_{\mathfrak{p}} \in K^\circ$ such that $\mathfrak{A}^{m_G \sigma_{\mathfrak{p}} - N(\mathfrak{p})\theta_{K/k,S}} = \alpha_{\mathfrak{p}} \mathcal{O}_K$ and $\alpha_{\mathfrak{p}} \equiv 1 \pmod{\mathcal{O}_K}$ for all $\mathfrak{p} | p$ with $\sigma_{\mathfrak{p}} = \sigma_{\mathfrak{p}}$.

3. For all $H < G$, abelian, there exist a family $(a_i)_{i \in I}$ of element of $\mathbb{Z}[H]$ generating $\text{Ann}_{\mathbb{Z}[H]}(\mu_K)$ and a family $(\alpha_i)_{i \in I} \subset K^\circ$ such that $\mathfrak{A}^{m_G a_i \theta_{K/k,S}} = \alpha_i \mathcal{O}_K$ and $\alpha_i^{a_j} = \alpha_j^{a_i}$ for all $i, j \in I$.

The set $\{\mathfrak{A} : \text{BS}_{\text{Gal}}(K/k, S; \mathfrak{A}) \text{ is true}\}$ is a group stable under the action of $G$ that contains the principal ideals of $K$. 
A Brumer-Stark conjecture for Galois extensions

The local Galois Brumer-Stark conjecture: the local conjecture

The local Galois Brumer-Stark Conjecture $\text{BS}_{\text{Gal}}^{(\ell)}(K/k, S)$

Let $w_K, \ell$ be the $\ell$-part of $w_K$. We have

$$m_G w_K \theta_{K/k, S} \in \text{Ann}_{\mathbb{Z}[G]}(\text{Cl}_K \{\ell\})$$

Furthermore, for $\mathcal{A}$ with $[\mathcal{A}] \in \text{Cl}_K \{\ell\}$, there exists $\alpha \in K^\times$ with

- $\mathcal{A}^{m_G w_K \theta_{K/k, S}} = (\alpha)$
- $\alpha \in K^\circ$
- $K(\alpha^{1/w_K, \ell})$ is a strong central extension of $K/k$

We have

$$\text{BS}_{\text{Gal}}(K/k, S) \iff \text{BS}_{\text{Gal}}^{(\ell)}(K/k, S) \quad \forall \ell$$
Consider $K'/k$ a Galois sub-extension of $K/k$ with group $G'$.

Let $\ell$ such that one at least of the following conditions is true:

- $\ell \nmid w_K$,
- $m_G w_K \theta_{K'/k}, S \notin \ell\mathbb{Z}[G']$,
- there is no abelian sub-extension of $K/K'K^{ab}$ of degree $\ell$ unramified outside of $w_K$.

Then

$$\text{BS}^{(\ell)}_{\text{Gal}}(K/k, S) \implies \tilde{\text{BS}}^{(\ell)}_{\text{Gal}}(K'/k, S)$$

where $\tilde{\text{BS}}^{(\ell)}_{\text{Gal}}(K'/k, S)$ is $\text{BS}^{(\ell)}_{\text{Gal}}(K'/k, S')$ with $m_{G'}$ replaced by $m_G$.

(One proves easily that $m_{G'}$ divides $m_G$.)
We assume that there exists $H < G$, abelian, with $(G : H) = p$.

We have

$$
\theta_{K/k,S} = \frac{1}{|[G, G]|} (\theta_{K^{ab}/k,S} N_{K/K^{ab}} - \theta_{K^{ab}/K^H, S_H} N_{K/K^{ab}}) + \theta_{K/K^H, S_H}
$$

where $S_H := \{ w \in \text{Pl}(K^H) : w|_k \in S \}$.

In this case, $|[G, G]| \mid m_G$, thus $m_G \xi \cdot \theta_{K/k,S} \in \mathbb{Z}[G]$, for all $\xi \in \text{Ann}_{\mathbb{Z}[G]}(\mu(K))$. 
A Brumer-Stark conjecture for Galois extensions

The Galois Brumer-Stark conjecture: abelian subgroup of prime index (odd case)

Recall that

\[ \theta_{K/k,S} = \frac{1}{|[G,G]|} (\theta_{K^{ab}/k,S} N_{K/K^{ab}} - \theta_{K^{ab}/K^H,S_H} N_{K/K^{ab}}) + \theta_{K/K^H,S_H} \]

If \( H \) is of odd order then \( K^H \) is not totally real and

\[ \theta_{K^{ab}/K^H,S_H} = \theta_{K/K^H,S_H} = 0. \]

We prove that \( \text{BS}(K^{ab}/k,S) \iff \text{BS}_{\text{Gal}}(K/k,S) \).

Consequence. \( \text{BS}_{\text{Gal}}(K/k,S) \) is true if \( G \simeq D_n \) with \( n \) odd.
A Brumer-Stark conjecture for Galois extensions

The Galois Brumer-Stark conjecture: abelian subgroup of prime index (even case)

Recall that

\[ \theta_{K/k,S} = \frac{1}{[[G, G]]} \left( \theta_{K^{ab}/k, SN_{K/K^{ab}}}^{K^{ab}/K^{ab}, S_{H^{K/K^{ab}}}} - \theta_{K^{ab}/K^{H}, S_{H^{K/K^{ab}}}} \right) + \theta_{K/K^{H}, S_{H}} \]

Assume \( H \) is of even order and that \( BS(K^{ab}/k, S) \) and \( BS(K/K^{H}, S_{H}) \) hold (thus also \( BS(K^{ab}/K^{H}, S) \)). Then

\[ m_{G} w_{K} \theta_{K/k,S} \in \text{Ann}_{\mathbb{Z}[G]}(\text{Cl}_{K}) \]

And, for any fractional ideal \( \mathcal{A} \) of \( K \), there exists \( \alpha \in K^{\times} \) with

- \( \mathcal{A}^{m_{G} w_{K} \theta_{K/k,S}} = (\alpha) \)
- \( \alpha \in K^{\circ} \)
- \( K(\alpha^{1/w_{K}})/K^{H} \) is abelian

**Consequence.** \( BS_{\text{Gal}}(K/k, S) \) is true if \( G \) is non-abelian of order 8.