Atelier PARI

Modular parametrization

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$(\mathrm{Lm}^{\mathrm{B}})$

18 janvier 2013

Modular parametrization

Let E be an elliptic curve defined over $\mathbb{Q}$ with conductor N.

- ▷ There exists a map

$$\varphi : X_0(N) \longrightarrow E$$

- ▷ It is (nearly) explicit

$$\begin{array}{cccccc} \varphi: & X_0(N)(\mathbb{C}) & \longrightarrow & \mathbb{C}/\Lambda & \longrightarrow & E(\mathbb{C}) \\ & \tau \in \mathbb{H} & \longmapsto & z = c\sum_{n\geq 1} \frac{a(n)}{n} e^{2i\pi n\tau} & \longmapsto & (\wp(z), \wp'(z)) \end{array}$$

where :

- $L(E,s) = \sum_{n\geq 1} \dfrac{a(n)}{n^s}$ is the L-function associated to E;
- c is Manin's constant of E.

Problem. Compute $\deg(\varphi)$.

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Why to compute $\deg \varphi$?

- It is a natural invariant attached to E;
- the primes dividing $\deg(\varphi)$ have certain properties;
- the growth of $\deg(\varphi)$ is link with certain conjecture;
- links with the Petersson norm of f the weight 2 modular form associated to E.
- ...

Example. $E\colon y^2 = x^3 + 11x + 13$, $N = 39548$.

$$\deg(\varphi) = 5376 = 2^8 \times 3 \times 7.$$

Conjecture (M. Watkins)

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▷ If E_d is a quadratic twist of E, $\deg(\varphi_E)$ and $\deg(\varphi_{E_d})$ are related.

→ Assume E is "minimal" among all its twists (conductor and disc.).

→ Compute this minimal curve.

▷ We use (Theorem of Zagier)

$$\deg(\varphi) = \frac{Nc^2}{2\pi \operatorname{vol}(\Lambda)} L(\operatorname{sym}^2 E, 2) \prod_{p^2 \mid N} L_p(\operatorname{sym}^2 E, p^{-2})$$

where

$$L(\operatorname{sym}^2 E, s) = \frac{\zeta_N(2s-2)}{\zeta_N(s-1)} \left(\sum_n \frac{a(n)^2}{n^s} \right) \prod_{p^2 \mid N} L_p(\operatorname{sym}^2 E, p^{-s})^{-1}, \Re(s) > 2.$$

We need to determine the conductor B of $L(\operatorname{sym}^2 E, s)$ and the Euler factor $L_p(\operatorname{sym}^2 E, X)$ for $p^2 \mid N$.

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B and $L_p(\mathrm{sym}^2 E, X)$

▷ If $p \nmid N$ or $p \| N$ then

$$\mathrm{val}_p(B) = \mathrm{val}_p(N) \quad \text{and } L_p(\mathrm{sym}^2 E, X) = (1-\alpha_p^2 X)(1-\alpha_p\beta_p X)(1-\beta_p^2 X)$$

where α_p and β_p are the roots of $X^2 - a(p)X + p$.

▷ If $p^2 | N$, then $L_p(\mathrm{sym}^2 E, X) = 1 + \varepsilon p X$ with $\varepsilon = -1, 0, +1$.

For example, thanks to M. Watkins: if

$p \equiv 1 \mod 12)$ or

$p \equiv 5 \mod 12$ and $p^2 \mid c_6$ and $p^2 \nmid c_4$ or

$p \equiv 7 \mod 12$ and $p^2 \nmid c_6$ or $p^2 \mid c_6$ and $p^2 \mid c_4$

then $\varepsilon = -1$ and $\mathrm{val}_p(B) = 1$.

▷ There are other technical but explicit rules for the other primes.

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The primitive square L-function

Theorem (Coates-Schmidt)

The function $L(\mathrm{sym}^2 E, s)$ has a holomorphic continuation to the whole complex plane and the function:

$$\Lambda(\mathrm{sym}^2 E, s) = \left(\frac{B}{2\pi^{3/2}}\right)^s \Gamma(s)\Gamma(s/2)L(\mathrm{sym}^2 E, s)$$

is entire and satisfies the functional equation

$$\Lambda(\mathrm{sym}^2 E, s) = \Lambda(\mathrm{sym}^2 E, 3-s).$$

$\to$ Just have to compute $\Lambda(\mathrm{sym}^2 E, 2)$ using the classical machinery.

Remarks.

We have $L(\mathrm{sym}^2 E, s) = L(\mathrm{sym}^2 E_d, s)$.

If $L(\mathrm{sym}^2 E, s) = \sum \frac{b(n)}{n^s}$ then $\sum_n \frac{b(n)}{n^2}$ is (slowly) converging.

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Computing $\Lambda(\mathrm{sym}^2 E, s) = \gamma(s) L(\mathrm{sym}^2 E, s)$

We have

$$\Lambda(\mathrm{sym}^2 E, s) = \sum_{n=1}^{N_0} \frac{b(n)}{n^s} F(s,n) + \sum_{n=1}^{N_0} \frac{b(n)}{n^{3-s}} F(3-s,n) + Error,$$

where

$$F(s,x) = \gamma(s) - \int_0^x \frac{1}{2i\pi} \int_{\Re(z)=\delta} t^{s-z-1} \gamma(s) dz dt.$$

And

$$|F(s,x)| \leq 7 \frac{x^{\Re(s)}}{A - \Re(s) A^{1/3}} e^{3/2 A^{2/3}}$$

for $A = \frac{2^{3/4} \pi^{3/2} x}{B}$.

$\rightarrow$ Useful for computing N_0 in function of the $Error$.

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with

$$\begin{aligned} u_{2q} &= \frac{2(-1)^q}{C^{2q} q!(2q)!}; \\ u_{2q+1} &= \frac{(-1)^q \sqrt{\pi} 2^{2q+1} q!}{(2q+1)!^2 C^{2q+1}}; \\ v_{2q} &= \frac{2(-1)^q}{C^{2q} q!(2q)!} \left(\log(C) - \frac{3}{2}\gamma \frac{1}{2} \sum_{j=1}^{q} j^{-1} + \sum_{j=1}^{2q} j^{-1} \right). \end{aligned}$$

Where $C = \frac{B}{2\pi^{3/2}}$.

▷ Need to determine i_0 (depends on x and s).

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Computing $F(s,x)$

We have

$$F(s,x) = \gamma(s) - \sum_{q\geq 0}^{i_0} x^{s+2q}\left(\frac{v_{2q} - u_{2q}\log(x)}{s+2q} + \frac{u_{2q}}{(s+2q)^2} + \frac{x u_{2q+1}}{s+2q+1}\right),$$

with

$$\begin{aligned} u_{2q} &= \frac{2(-1)^q}{C^{2q} q!(2q)!}; \\ u_{2q+1} &= \frac{(-1)^q \sqrt{\pi} 2^{2q+1} q!}{(2q+1)!^2 C^{2q+1}}; \\ v_{2q} &= \frac{2(-1)^q}{C^{2q} q!(2q)!}\left(\log(C) - \frac{3}{2}\gamma\frac{1}{2}\sum_{j=1}^{q} j^{-1} + \sum_{j=1}^{2q} j^{-1}\right). \end{aligned}$$

Where $C = \frac{B}{2\pi^{3/2}}$.

▷ Need to determine i_0 (depends on x and s).

Summary

▷ $\deg(\varphi) = \frac{Nc^2}{2\pi \operatorname{vol}(\Lambda)} L(\operatorname{sym}^2 E, 2) \prod_{p^2|N} L_p(\operatorname{sym}^2 E, p^{-2})$.

▷ It is an integer:

→ so we need to compute $L(\operatorname{sym}^2 E, 2)$ up to a certain $error$.

→ this gives a check for the computation.

→ it can be large!

→ **Question.** $\deg(\varphi)$ is an integer.

Is it possible to compute $\deg(\varphi) \mod \ell$ for many primes ℓ?

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Problems to be fixed

$\triangleright$ Compute Manin's constant (hard!);

$\triangleright$ Formula not numerically stable for $F(s, x)$ (many cancellation problems, not so easy to fix...);

$\triangleright$ The value of N_0 is not computed efficiently (easy to fix).

$\triangleright$ Need a more clever and efficient management with quadratic twists (easy to fix).