Symbolic integration in finite chararcteristic

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Elementary functions

Traditionnaly, a function from $\mathbb{C}$ to $\mathbb{C}$ is said to be *elementary* if it can be written as a composite of multivariate algebraic functions, the exponential and the logarithm.
Liouville in 1838 showed that some elementary functions do not admit an elementary antiderivative. Two important examples are $x \mapsto \exp(x)/x$ and $x \mapsto \sqrt{x^4+1}$. Risch in 1968 gives an algorithm to decide if an elementary function admit an elementary derivative.

Derivations

Ritt in 1930 gives an algebraic definition of elementary functions in term of differential fields which do not require functions to be composable, or even to be have values at points.

Let K be a field. A *derivation* on K is a group homomorphism $\partial = \partial_K$ from K to K such that for all a, b in K the following holds:

$$\partial(ab) = \partial(a)b + a\partial(b) \quad .$$

Example: if K is a field, then $f \mapsto f'$ is a derivation over $K(X)$.

Differential fields

A couple (K, ∂_K) is a *differential field* if K is a field and ∂_K is a derivation on K.
A *differential extension* of (K, ∂_K) is a differential field (L, ∂_L) such that L/K is a field extension and $\partial_L|_K = \partial_K$.
The kernel of ∂_K is a field called the field of constants of (K, ∂_K), and is denoted by $C(K)$.

Elementary extensions

Theorem

If (K, ∂_K) is a differential field and L/K is an algebraic separable extension, then ∂_K can be extended in a unique way to a derivation ∂_L on L so that (L, ∂_L) is a differential extension of (K, ∂_K).

Elementary extensions

Definition

A differential extension (L, ∂_L) of (K, ∂_K) is logarithmic of type "$log(u)$" if there exists $u \in K$ and $y \in L$ generating L over K such that $\partial(u) = \partial(y)u$.

Definition

A differential extension (L, ∂_L) of (K, ∂_K) is exponential of type "$exp(u)$" if there exists $u \in K$ an $y \in L$ generating L over K such that $\partial(y) = \partial(u)y$.

Elementary extensions

Definition

A differential extension (L, ∂_L) of (K, ∂_K) is elementary if there exists a tower of differential extensions $L_0 = K \subseteq L_1 \ldots \subseteq L_n = L$ such that each extension L_{i+1}/L_i is either algebraic separable, logarithmic, or exponential.

Traditional elementary functions are just elements of elementary extensions of $\mathbb{C}(X)$.

Differential fields in finite characteristic

Let (K, ∂_K) be a differential field of characteristic p, then

- for all $a \in K$, $\partial(a^p) = 0$ (i.e. $K^p \subseteq C(K)$).
- for all positive integers e, ∂^{p^e} is a derivation.

Definition (p-polynomials)

A polynomial P over a field K is a p-polynomial if it can be written as $P = \sum_{j=0}^{n} a_j X^{p^j}$ with $a_j \in K$ for $0 \leq j \leq n$.

If P is a p-polynomial, then $P(\partial_K)$ is a derivation on K.

Symbolic integration in finite characteristic

The goal of this talk is to prove

Theorem

Let k be a field of characteristic p, (K, ∂_K) an elementary extension of $(k(X), F- > F')$, f an element of K, then there exists a differential extension of logarithmic type (L, ∂_L) of (K, ∂_K) and $g \in L$ such that $\partial_L(g) = f$.

Linear differential equations

Definition

Let (K, ∂_K) be a differential field, let y be an element of K, and let C be a sub-differential field of K. We will say that $y \in K$ satisfies a linear differential equation with coefficients in C if there exists a non-zero polynomial $P \in C[X]$ such that $P(\partial)(y) = 0$.

Definition

A differential field (K, ∂_K) is *linear* if every element satisfies a linear differential equation with coefficients in $C(K)$.

p-polynomials

Lemma

If K is a field and I a non-zero ideal of $K[X]$, then I contains a non zero p-polynomial.

(follow from $K[X]/I$ being finite dimensionnal).

Lemma

Let (K, ∂_K) be a differential field, and $(y_i)_{i=1}^n$ be a family of elements of K. If the y_i satisfy a linear differential equation with constant coefficients for all $1 \leq i \leq n$, then there exists a non-zero p-polynomial P in $C(K)[X]$ such that $P(\partial)(y_i) = 0$ for all $1 \leq i \leq n$.

p-polynomials

Proposition

Let (K, ∂_K) be a differential field. The set $Lin(K)$ of elements of K that satisfy a linear differential equation with coefficients in $C(K)$ is a sub-differential field of K.

This follows from the fact that $C(P(\partial))$ is a field if P is a p-polynomial.

Elementary extensions are linear

Theorem

Let (K, ∂_K) be a linear differential field, and let (L, ∂_L) be a differential extension of (K, ∂_K). If L/K is elementary, then (L, ∂_L) is linear.

Lemma

The derivatives of the classical differential equation $y' = u'y$ are given by the multivariate Bell polynomials. In characteristic p, Carlitz Formula (1.4) gives the equation

$$\partial^{p^r}(y) = \left(\sum_{i=0}^{r} \partial^{p^i}(u)^{p^{r-i}}\right)y \quad . \tag{1}$$

Example

Let $K = (\mathbb{F}_p(E), \partial_K)$ be such that $\partial_K(E) = E$, and let L be the differential extension $(\mathbb{F}_p(E, F), \partial_L)$ where $\partial_L(F) = EF$. We find that $\partial_L^p(F) - \partial_L(F) = E^p F$.

Integrabe fields

Lemma

Let $(L, \partial)/(K, \partial)$ be a differential field of characteristic $p > 0$. If $u \in K$ is such that $\partial^{p^k}(u) \in C(K)^\times$ and $y \in L$ is such that $\partial(y) = \partial(u)/u$, then $\partial^{p^{k+1}}(y) \in C(L)^\times$.

Proposition

Let (K, ∂) be a differential field of characteristic $p > 0$, and assume that there exists $X \in K$ such that $\partial(X) = 1$. Then for all integers n there exist an elementary differential extension L/K and an element $y \in L$ such that $\partial^n(y) = 1$.

Integrable fields

Example

Let $p = 2$ we find for

$$\begin{array}{ll} n = 1 & x \\ n = 2 & x^2 \log(x) \\ n = 4 & x^4 \log(x)^2 \log(\log(x)) \\ n = 8 & x^8 \log(x)^2 \log(\log(x))^4 \log(\log(\log(x))) \end{array}$$

with obvious notation.

Integrable fields

Definition

A differential field (K, ∂) is *integrable* if it is linear and ∂ takes the value 1 on K.

Example

The differential field $(\mathbb{F}_p(X), \partial)$ where ∂ is the standard derivation is integrable.
Indeed, every element u satisfies $\partial^p(u) = 0$, and $\partial(X) = 1$.

Theorem

Every elementary extension of an integrable differential field is integrable.

Integrable fields

Lemma

Let (K, ∂) be a differential field. If u and v in K and $n \geq 1$ are such that $\partial^n(u) = 0$ and $\partial^{n-1}(v) = 1$, then u belongs to the vector space generated by $(\partial^k(v))_{k=0}^{n-1}$ over $C(K)$.

Theorem

Let (K, ∂) be an integrable differential field. Every element $u \in K$ admits an antiderivative in an extension of logarithmic type.

Integrable fields

Example

We look for the characterictic p analogue of the antiderivative of $exp(exp(x))$. From $\partial^p(F) - \partial(F) = E^p F$, we conclude that $\frac{\partial^{p-1}(F)-F}{E^p}$ is an antiderivative of F.

Linear differential equations

Theorem

Let (K, ∂) be an integrable differential field and $Q \in C(K)[X]$. The linear differential equation $Q(\partial)(y) = 0$ admits a solution in some elementary extension of K.

Theorem

Let (K, ∂_K) be a linear differential field, and (L, ∂_L) a differential extension of (K, ∂_K). If $y \in L$ statisfies a linear differential with coefficients in K then it satisfies a linear differential equation with coefficients in $C(K)$.

Linear differential equations

Theorem

Let (K, ∂) be an integrable differential field and $P \in K[X]$. The linear differential equation $P(\partial)(y) = 0$ admits a solution in some elementary extension of K.