

# Elliptic Curves

(PARI-GP version 2.18.1)

An elliptic curve is initially given by 5-tuple  $v = [a_1, a_2, a_3, a_4, a_6]$  attached to Weierstrass model or simply  $[a_4, a_6]$ . It must be converted to an *ell* struct.

Initialize *ell* struct over domain  $D$      **E** = `ellinit(v, {D = 1})`  
over  $\mathbf{Q}$       $D = 1$   
over  $\mathbf{F}_p$       $D = p$   
over  $\mathbf{F}_q$ ,  $q = p^f$       $D = \text{ffgen}([p, f])$   
over  $\mathbf{Q}_p$ , precision  $n$       $D = O(p^n)$   
over  $\mathbf{C}$ , current bitprecision      $D = 1.0$   
over number field  $K$       $D = nf$

Points are  $[x,y]$ , the origin is  $[0]$ . Struct members accessed as **E.member**:

- All domains: **E.a1,a2,a3,a4,a6, b2,b4,b6,b8, c4,c6, disc, j**
- **E** defined over  $\mathbf{R}$  or  $\mathbf{C}$   
 $x$ -coords. of points of order 2     **E.roots**  
periods / quasi-periods     **E.omega, E.eta**  
volume of complex lattice     **E.area**
- **E** defined over  $\mathbf{Q}_p$   
residual characteristic     **E.p**  
If  $|j|_p > 1$ : Tate's  $[u^2, u, q, [a, b], \mathcal{L}]$      **E.tate**
- **E** defined over  $\mathbf{F}_q$   
characteristic     **E.p**  
 $\#E(\mathbf{F}_q)/\text{cyclic structure/generators}$      **E.no, E.cyc, E.gen**
- **E** defined over  $\mathbf{Q}$   
generators of  $E(\mathbf{Q})$  (require `elldata`)     **E.gen**  
 $[a_1, a_2, a_3, a_4, a_6]$  from  $j$ -invariant     `ellfromj(j)`  
cubic/quartic/biquadratic to Weierstrass     `ellfromeqn(eq)`  
add points  $P + Q / P - Q$      `elladd(E, P, Q), ellsub`  
negate point     `ellneg(E, P)`  
compute  $n \cdot P$      `ellmul(E, P, n)`  
sum of Galois conjugates of  $P$      `elltrace(E, P)`  
check if  $P$  is on  $E$      `ellisoncurve(E, P)`  
order of torsion point  $P$      `ellorder(E, P)`  
 $y$ -coordinates of point(s) for  $x$      `ellordinate(E, x)`  
 $[\phi(z), \phi'(z)] \in E(\mathbf{C})$  attached to  $z \in \mathbf{C}$      `ellztpoint(E, z)`  
 $z \in \mathbf{C}$  such that  $P = [\phi(z), \phi'(z)]$      `ellpointtoz(E, P)`  
 $z \in \bar{\mathbf{Q}}^*/q^{\mathbf{Z}}$  to  $P \in E(\bar{\mathbf{Q}}_p)$      `ellztopoint(E, z)`  
 $P \in E(\bar{\mathbf{Q}}_p)$  to  $z \in \bar{\mathbf{Q}}^*/q^{\mathbf{Z}}$      `ellpointtoz(E, P)`
- **Change of Weierstrass models, using**  $v = [u, r, s, t]$   
change curve  $E$  using  $v$      `ellchangecurve(E, v)`  
change point  $P$  using  $v$      `ellchangept(E, P, v)`  
change point  $P$  using inverse of  $v$      `ellchangeptinv(P, v)`  
is  $E$  isomorphic to  $F$ ?     `ellisism(E, F)`
- **Twists and isogenies**  
quadratic twist     `elltwist(E, d)`  
 $n$ -division polynomial  $f_n(x)$      `elldivpol(E, n, {x})`  
 $[n]P = (\phi_n \psi_n : \omega_n : \psi_n^3)$ ; return  $(\phi_n, \psi_n^2)$      `ellxn(E, n, {x})`  
isogeny from  $E$  to  $E/G$      `ellisogeny(E, G)`  
apply isogeny to  $g$  (point or isogeny)     `ellisogenyapply(f, g)`  
torsion subgroup with generators     `elltors(E)`
- **Formal group**  
formal exponential,  $n$  terms     `ellformalexp(E, {n}, {x})`  
formal logarithm,  $n$  terms     `ellformallog(E, {n}, {x})`  
 $\log_E(-x(P)/y(P)) \in \mathbf{Q}_p$ ;  $P \in E(\mathbf{Q}_p)$      `ellpadiclog(E, p, n, P)`  
 $P$  in the formal group     `ellformalpoint(E, {n}, {x})`  
 $[w/dt, xw/dt]$      `ellformaldifferential(E, {n}, {x})`

$w = -1/y$  in parameter  $-x/y$      `ellformalw(E, {n}, {x})`

## Curves over finite fields, Pairings

random point on  $E$      `random(E)`  
 $\#E(\mathbf{F}_q)$      `ellcard(E)`  
 $\#E(\mathbf{F}_q)$  with almost prime order     `ellsea(E, {tors})`  
structure  $\mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z}$  of  $E(\mathbf{F}_q)$      `ellgroup(E)`  
is  $E$  supersingular?     `ellissupersingular(E)`  
random supersingular  $j$ -invariant over  $F_2^2$      `ellsupersingularj(p)`  
Weil pairing of  $m$ -torsion pts  $P, Q$      `ellweilpairing(E, P, Q, m)`  
Tate pairing of  $P, Q$ ;  $P$   $m$ -torsion     `elltatepairing(E, P, Q, m)`  
Discrete log, find  $n$  s.t.  $P = [n]Q$      `elllog(E, P, Q, {ord})`

## Curves over $\mathbf{Q}$

### Reduction, minimal model

minimal model of  $E/\mathbf{Q}$      `ellminimalmodel(E, {\&v})`  
quadratic twist of minimal conductor     `ellminimaltwist(E)`  
 $[k]P$  with good reduction     `ellnonsingularmultiple(E, P)`  
 $E$  supersingular at  $p$ ?     `ellissupersingular(E, p)`  
affine points of naive height  $\leq h$      `ellratpoints(E, h)`

### Complex heights

canonical height of  $P$      `ellheight(E, P)`  
canonical bilinear form taken at  $P, Q$      `ellheight(E, P, Q)`  
height regulator matrix for pts in  $L$      `ellheightmatrix(E, L)`

### $p$ -adic heights

cyclotomic  $p$ -adic height of  $P \in E(\mathbf{Q})$      `ellpadicheight(E, p, n, P)`  
 $\dots$  bilinear form at  $P, Q \in E(\mathbf{Q})$      `ellpadicheight(E, p, n, P, Q)`  
 $\dots$  matrix at vector for pts in  $L$      `ellpadicheightmatrix(E, p, n, L)`  
 $\dots$  regulator for canonical height     `ellpadicregulator(E, p, n, Q)`  
Frobenius on  $\mathbf{Q}_p \otimes H_{dR}^1(E/\mathbf{Q})$      `ellpadicfrobenius(E, p, n)`  
slope of unit eigenvector of Frobenius     `ellpadics2(E, p, n)`

### Isogenous curves

matrix of isogeny degrees for  $\mathbf{Q}$ -isog. curves     `ellisomat(E)`  
tree of prime degree isogenies     `ellisotree(E)`  
a modular equation of prime degree  $N$      `ellmodulareqn(N)`

### $L$ -function

$p$ -th coeff  $a_p$  of  $L$ -function,  $p$  prime     `ellap(E, p)`  
 $k$ -th coeff  $a_k$  of  $L$ -function     `ellak(E, k)`  
 $L(E, s)$  (using less memory than `lfun`)     `elllseries(E, s)`  
 $L^{(r)}(E, 1)$  (using less memory than `lfun`)     `ellL1(E, r)`  
a Heegner point on  $E$  of rank 1     `ellheegner(E)`  
order of vanishing at 1     `ellanalyticrank(E, {eps})`  
root number for  $L(E, \cdot)$  at  $p$      `ellrootno(E, {p})`  
modular parametrization of  $E$      `elltaniyama(E)`  
degree of modular parametrization     `ellmoddegree(E)`  
compare with  $H^1(X_0(N), \mathbf{Z})$  (for  $E' \rightarrow E$ )     `ellweilcurve(E)`  
Manin constant of  $E$      `ellmaninconstant(E)`

$p$ -adic  $L$  function  $L_p^{(r)}(E, d, \chi^s)$      `ellpadicl(E, p, n, {s}, {r}, {d})`  
BSD conjecture for  $L_p^{(r)}(E_D, \chi^0)$      `ellpadicbsd(E, p, n, {D = 1})`  
Iwasawa invariants for  $L_p(E_D, \tau^i)$      `ellpadiclambda(E, p, D, i)`

### Rational points

attempt to compute  $E(\mathbf{Q})$      `ellrank(E, {effort}, {points})`  
initialize for later `ellrank` calls,     `ellrankinit(E)`  
saturate  $\langle P_1, \dots, P_n \rangle$  wrt. primes  $\leq B$      `ellsaturation(E, P, B)`  
2-covers of the curve  $E$      `ell2cover(E)`

### Elldata package, Cremona's database:

db code "11a1"  $\leftrightarrow$  `[conductor, class, index]`     `ellconvertname(s)`  
generators of Mordell-Weil group     `ellgenerators(E)`  
look up  $E$  in database     `ellidentify(E)`

all curves matching criterion     `ellsearch(N)`  
loop over curves with cond. from  $a$  to  $b$      `forell(E, a, b, seq)`

## Curves over number field $K$

coeff  $a_p$  of  $L$ -function     `ellap(E, p)`  
Kodaira type of  $\mathfrak{p}$ -fiber of  $E$      `elllocalred(E, p)`  
integral model of  $E/K$      `ellintegralmodel(E, {\&v})`  
minimal model of  $E/K$      `ellminimalmodel(E, {\&v})`  
minimal discriminant of  $E/K$      `ellminimaldisc(E)`  
cond, min mod, Tamagawa num  $[N, v, c]$      `ellglobalred(E)`  
global Tamagawa number     `elltamagawa(E)`  
test if  $E$  has CM     `elliscm(E)`  
 $P \in E(K)$   $n$ -divisible?  $[n]Q = P$      `ellisdivisible(E, P, n, {\&Q})`

### $L$ -function

A domain  $D = [c, w, h]$  in initialization mean we restrict  $s \in \mathbf{C}$  to domain  $|\Re(s) - c| < w$ ,  $|\Im(s)| < h$ ;  $D = [w, h]$  encodes  $[1/2, w, h]$  and  $[h]$  encodes  $D = [1/2, 0, h]$  (critical line up to height  $h$ ).  
vector of first  $n$   $a_k$ 's in  $L$ -function     `ellan(E, n)`  
init  $L^{(k)}(E, s)$  for  $k \leq n$      **L** = `lfuninit(E, D, {n = 0})`  
compute  $L(E, s)$  ( $n$ -th derivative)     `lfun(L, s, {n = 0})`  
 $L(E, 1, r)/(r! \cdot R \cdot \#Sha)$  assuming BSD     `ellbsd(E)`

## Other curves of small genus

A hyperelliptic curve  $C$  is given by a pair  $[P, Q]$  ( $y^2 + Qy = P$  with  $Q^2 + 4P$  squarefree) or a single squarefree polynomial  $P$  ( $y^2 = P$ ).  
check if  $[x, y]$  is on  $C$      `hyperellisoncurve(C, [x, y])`  
 $y$ -coordinates of point(s) for  $x$      `hyperellordinate(C, x)`  
discriminant of  $C$      `hyperelldisc(C)`  
Cremona-Stoll reduction     `hyperellred(C)`  
apply  $m = [e, [a, b; c, d], H]$  to model     `hyperellchangecurve(C, m)`  
minimal discriminant of integral  $C$      `hyperellminimaldisc(C)`  
minimal model of integral  $C$      `hyperellminimalmodel(C)`  
reduction of  $y^2 + Qy = P$  (genus 2)     `genus2red(C, {p})`  
Igusa invariants for  $C$  of genus 2     `genus2igusa(C)`  
affine rational points of height  $\leq h$      `hyperellratpoints(C, h)`  
find a rational point on a conic,  ${}^tXGX = 0$      `qfsolve(G)`  
 $[H, U]$  such that  $H = c^tUGU$  has minimat `d`     `qfminimize(G)`  
quadratic Hilbert symbol (at  $p$ )     `hilbert(x, y, {p})`  
all solutions in  $\mathbf{Q}^3$  of ternary form     `qfparam(G, x)`  
 $P, Q \in \mathbf{F}_q[X]$ ; char. poly. of Frobenius     `hyperellcharpoly(Q)`  
matrix of Frobenius on  $\mathbf{Q}_p \otimes H_{dR}^1$      `hyperellpadicfrobenius`

## Elliptic & Modular Functions

$w = [\omega_1, \omega_2]$  or *ell* struct (**E.omega**),  $\tau = \omega_1/\omega_2$ .  
arithmetic-geometric mean     `agm(x, y)`  
elliptic  $j$ -function  $1/q + 744 + \dots$      `ellj(x)`  
Weierstrass  $\sigma/\wp/\zeta$  function     `ellsigma(w, z), ellwp, ellzeta`  
periods/quasi-periods     `ellperiods(E, {flag}), elleta(w)`  
 $(2i\pi/\omega_2)^k E_k(\tau)$      `elleisnum(w, k, {flag})`  
modified Dedekind  $\eta$  func.  $\prod(1 - q^n)$      `eta(x, {flag})`  
Dedekind sum  $s(h, k)$      `sumdedekind(h, k)`  
Jacobi sine theta function     `theta(q, z)`  
 $k$ -th derivative at  $z=0$  of `theta(q, z)`     `thetanullk(q, k)`  
Weber's  $f$  functions     `weber(x, {flag})`  
modular pol. of level  $N$      `polmodular(N, {inv = j})`  
Hilbert class polynomial for  $\mathbf{Q}(\sqrt{D})$      `polclass(D, {inv = j})`

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