

Elliptic Curves

(PARI-GP version 2.12.0)

An elliptic curve is initially given by 5-tuple $v = [a_1, a_2, a_3, a_4, a_6]$ attached to Weierstrass model or simply $[a_4, a_6]$. It must be converted to an *ell* struct.

Initialize *ell* struct over domain D **E = ellinit**($v, \{D = 1\}$)
 over **Q** $D = 1$
 over **F_p** $D = p$
 over **F_q**, $q = p^f$ $D = \text{ffgen}([p, f])$
 over **Q_p**, precision n $D = O(p^n)$
 over **C**, current bitprecision $D = 1.0$
 over number field K $D = nf$

Points are $[x, y]$, the origin is $[0]$. Struct members accessed as **E.member**:

- All domains: **E.a1, a2, a3, a4, a6, b2, b4, b6, b8, c4, c6, disc, j**
- E defined over **R** or **C**
 x -coords. of points of order 2 **E.roots**
 periods / quasi-periods **E.omega, E.eta**
 volume of complex lattice **E.area**

- E defined over **Q_p**
 residual characteristic **E.p**
 If $|j|_p > 1$: Tate's $[u^2, u, q, [a, b], \mathcal{L}]$ **E.tate**
- E defined over **F_q**
 characteristic **E.p**
 $\#E(\mathbf{F}_q)/\text{cyclic structure/generators}$ **E.no, E.cyc, E.gen**

- E defined over **Q**
 generators of $E(\mathbf{Q})$ (require **elldata**) **E.gen**
 $[a_1, a_2, a_3, a_4, a_6]$ from j -invariant **ellfromj(j)**
 cubic/quartic/biquadratic to Weierstrass **ellfromeqn(eq)**
 add points $P + Q / P - Q$ **elladd(E, P, Q), ellsub**

- negate point **ellneg(E, P)**
- compute $n \cdot P$ **ellmul(E, P, n)**
- check if P is on E **ellisoncurve(E, P)**
- order of torsion point P **ellorder(E, P)**
- y -coordinates of point(s) for x **ellordinate(E, x)**
- $[\phi(z), \phi'(z)] \in E(\mathbf{C})$ attached to $z \in \mathbf{C}$ **ellztopoint(E, z)**
- $z \in \mathbf{C}$ such that $P = [\phi(z), \phi'(z)]$ **ellpointtoz(E, P)**
- $z \in \bar{\mathbf{Q}}^*/q\mathbf{Z}$ to $P \in E(\bar{\mathbf{Q}}_p)$ **ellztopoint(E, z)**
- $P \in E(\bar{\mathbf{Q}}_p)$ to $z \in \bar{\mathbf{Q}}^*/q\mathbf{Z}$ **ellpointtoz(E, P)**

- Change of Weierstrass models, using** $v = [u, r, s, t]$
 change curve E using v **ellchangecurve(E, v)**
 change point P using v **ellchangept(E, P, v)**
 change point P using inverse of v **ellchangeptinv(E, P, v)**

- Twists and isogenies**
 quadratic twist **elltwist(E, d)**
 n -division polynomial $f_n(x)$ **elldivpol(E, n, \{x\})**
 $[n]P = (\phi_n \psi_n, \omega_n, \psi_n^2)$; return (ϕ_n, ψ_n^2) **ellxn(E, n, \{x\})**
 isogeny from E to E/G **ellisogeny(E, G)**
 apply isogeny to g (point or isogeny) **ellisogenyapply(f, g)**
 torsion subgroup with generators **elltors(E)**

- Formal group**
 formal exponential, n terms **ellformalexp(E, \{n\}, \{x\})**
 formal logarithm, n terms **ellformallog(E, \{n\}, \{x\})**
 $\log_E(-x(P)/y(P)) \in \mathbf{Q}_p$; $P \in E(\mathbf{Q}_p)$ **ellpadiclog(E, p, n, P)**
 P in the formal group **ellformalpoint(E, \{n\}, \{x\})**
 $[\omega/dt, x\omega/dt]$ **ellformaldifferential(E, \{n\}, \{x\})**
 $w = -1/y$ in parameter $-x/y$ **ellformalw(E, \{n\}, \{x\})**

Curves over finite fields, Pairings

- random point on E **random(E)**
- $\#E(\mathbf{F}_q)$ **ellcard(E)**
- $\#E(\mathbf{F}_q)$ with almost prime order **ellsea(E, \{tors\})**
- structure $\mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z}$ of $E(\mathbf{F}_q)$ **ellgroup(E)**
- is E supersingular? **ellissupersingular(E)**
- Weil pairing of m -torsion pts P, Q **ellweilpairing(E, P, Q, m)**
- Tate pairing of P, Q ; P m -torsion **elltatepairing(E, P, Q, m)**
- Discrete log, find n s.t. $P = [n]Q$ **elllog(E, P, Q, \{ord\})**

Curves over Q

- Reduction, minimal model**
 minimal model of E/\mathbf{Q} **ellminimalmodel(E, \{\&v\})**
 quadratic twist of minimal conductor **ellminimaltwist(E)**
 $[k]P$ with good reduction **ellnonsingularmultiple(E, P)**
 E supersingular at p ? **ellissupersingular(E, p)**
 affine points of naïve height $\leq h$ **ellratpoints(E, h)**

- Complex heights**
 canonical height of P **ellheight(E, P)**
 canonical bilinear form taken at P, Q **ellheight(E, P, Q)**
 height regulator matrix for pts in L **ellheightmatrix(E, L)**

- p -adic heights**
 cyclotomic p -adic height of $P \in E(\mathbf{Q})$ **ellpadicheight(E, p, n, P)**
 \dots bilinear form at $P, Q \in E(\mathbf{Q})$ **ellpadicheight(E, p, n, P, Q)**
 \dots matrix at vector for pts in L **ellpadicheightmatrix(E, p, n, L)**
 \dots regulator for canonical height **ellpadicregulator(E, p, n, Q)**
 Frobenius on $\mathbf{Q}_p \otimes H_{dR}^1(E/\mathbf{Q})$ **ellpadicfrobienius(E, p, n)**
 slope of unit eigenvector of Frobenius **ellpadic2(E, p, n)**

- Isogenous curves**
 matrix of isogeny degrees for **Q**-isog. curves **ellisomat(E)**
 tree of prime degree isogenies **ellisotree(E)**
 a modular equation of prime degree N **ellmodulareqn(N)**

- L -function**
 p -th coeff a_p of L -function, p prime **ellap(E, p)**
 k -th coeff a_k of L -function **ellak(E, k)**
 $L(E, s)$ (using less memory than **lfun**) **elllseries(E, s)**
 $L^{(r)}(E, 1)$ (using less memory than **lfun**) **elll1(E, r)**
 a Heegner point on E of rank 1 **ellheegner(E)**
 order of vanishing at 1 **ellanalyticrank(E, \{eps\})**
 root number for $L(E, \cdot)$ at p **ellrootno(E, \{p\})**
 modular parametrization of E **elltanayama(E)**
 degree of modular parametrization **ellmoddegree(E)**
 compare with $H^1(X_0(N), \mathbf{Z})$ (for $E' \rightarrow E$) **ellweilcurve(E)**

- p -adic L function $L_p^{(r)}(E, d, \chi^s)$ **ellpadicL(E, p, n, \{s\}, \{r\}, \{d\})**
 BSD conjecture for $L_p^{(r)}(E_D, \chi^0)$ **ellpadicbsd(E, p, n, \{D = 1\})**

- Elldata package, Cremona's database:**
 db code "11a1" \leftrightarrow [*conductor, class, index*] **ellconvertname(s)**
 generators of Mordell-Weil group **ellgenerators(E)**
 look up E in database **ellidentify(E)**
 all curves matching criterion **ellsearch(N)**
 loop over curves with cond. from a to b **forell(E, a, b, seq)**

Curves over number field K

- coeff a_p of L -function **ellap(E, p)**
- Kodaira type of \mathfrak{p} -fiber of E **elllocalred(E, p)**
- integral model of E/K **ellintegralmodel(E, \{\&v\})**
- minimal model of E/K **ellminimalmodel(E, \{\&v\})**
- minimal discriminant of E/K **ellminimaldisc(E)**
- cond, min mod, Tamagawa num $[N, v, c]$ **ellglobalred(E)**
- global Tamagawa number **elltamagawa(E)**
- $P \in E(K)$ n -divisible? $[n]Q = P$ **ellisdivisible(E, P, n, \{\&Q\})**

L -function

- A domain $D = [c, w, h]$ in initialization mean we restrict $s \in \mathbf{C}$ to domain $|\Re(s) - c| < w, |\Im(s)| < h; D = [w, h]$ encodes $[1/2, w, h]$ and $[h]$ encodes $D = [1/2, 0, h]$ (critical line up to height h).
- vector of first n a_k 's in L -function **ellan(E, n)**
- init $L^{(k)}(E, s)$ for $k \leq n$ **L = lfunit(E, D, \{n = 0\})**
- compute $L(E, s)$ (n -th derivative) **lfun(L, s, \{n = 0\})**
- $L(E, 1, r)/(r! \cdot R \cdot \#Sha)$ assuming BSD **ellbsd(E)**

Other curves of small genus

- A hyperelliptic curve is given by a pair $[P, Q]$ ($y^2 + Qy = P$ with $Q^2 + 4P$ squarefree) or a single squarefree polynomial P ($y^2 = P$).
- reduction of $y^2 + Qy = P$ (genus 2) **genus2red([P, Q], \{p\})**
- affine rational points of height $\leq h$ **hyperellratpoints([P, Q], h)**
- find a rational point on a conic, ${}^t xGx = 0$ **qfsolve(G)**
- quadratic Hilbert symbol (at p) **hilbert(x, y, \{p\})**
- all solutions in \mathbf{Q}^3 of ternary form **qfparam(G, x)**
- $P, Q \in \mathbf{F}_q[X]$; char. poly. of Frobenius **hyperellcharpoly([P, Q])**
- matrix of Frobenius on $\mathbf{Q}_p \otimes H_{dR}^1$ **hyperellpadicfrobienius**

Elliptic & Modular Functions

- $w = [\omega_1, \omega_2]$ or *ell* struct (**E.omega**), $\tau = \omega_1/\omega_2$.
- arithmetic-geometric mean **agm(x, y)**
- elliptic j -function $1/q + 744 + \dots$ **ellj(x)**
- Weierstrass $\sigma/\wp/\zeta$ function **ellsigma(w, z), ellwp, ellzeta**
- periods/quasi-periods **ellperiods(E, \{flag\}), elleta(w)**
- $(2i\pi/\omega_2)^k E_k(\tau)$ **elleisnum(w, k, \{flag\})**
- modified Dedekind η func. $\prod(1 - q^n)$ **eta(x, \{flag\})**
- Dedekind sum $s(h, k)$ **sumdedekind(h, k)**
- Jacobi sine theta function **theta(q, z)**
- k -th derivative at $z=0$ of **theta(q, z)** **thetanullk(q, k)**
- Weber's f functions **weber(x, \{flag\})**
- modular pol. of level N **polmodular(N, \{inv = j\})**
- Hilbert class polynomial for $\mathbf{Q}(\sqrt{D})$ **polclass(D, \{inv = j\})**

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