Elliptic curves over number fields

B. Allombert

IMB
CNRS/Université de Bordeaux

22/06/2018

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement N° 676541

Elliptic curves construction

An elliptic curve given from its short

$$y^2 = x^3 + a_4 x + a_6$$

or long

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

Weierstrass equation is defined by

```
? E=ellinit([a4,a6]);
? E=ellinit([a1,a2,a3,a4,a6]);
```

Elliptic curves construction

It is possible to obtain the Weierstrass equation of the Jacobian of a genus 1 curve. For example, for an Edward curve $ax^2 + y^2 = 1 + dx^2y^2$:

```
? e = ellfromeqn(a*x^2+y^2 - (1+d*x^2*y^2))
%1 = [0, -a - d, 0, -4*d*a, 4*d*a^2 + 4*d^2*a]
```

It is also possible to obtain a Weierstrass equation from a j-invariant.

```
? e = ellfromj(3)
%1 = [0,0,0,15525,17853750]
? E = ellinit(e);
? E.j
%3 = 3
? E.disc
%4 = -15380288749596672
```

Elliptic curves over the rationals

We define the elliptic curve $y^2 + y = x^3 + x^2 - 2x$ over the field $\mathbb{Q}$.

```
? E   = ellinit([0,1,1,-2,0]);
? E.j
%39 = 1404928/389
? E.disc
%40 = 389
? N   = ellglobalred(E)[1]
%41 = 389
? tor = elltors(E) \\ trivial
%42 = [1,[],[]]
? lfunorderzero(E)
%43 = 2
```

Elliptic curves over the rationals

```
? G = ellgenerators(E) \\ with elldata
? G = [[-1,1],[0,0]];  \\ without elldata
%44 = [[-1, 1], [0, 0]]
```

We check the BSD conjecture for E.

```
? tam = elltamagawa(E)
%45 = 2
? reg = matdet(ellheightmatrix(E,G));
? bsd = (E.omega[1]*tam)*reg
%46 = 0.75931650028842677023019260789472201908
? ellbsd(E)*reg
%47 = 0.75931650028842677023019260789472201908
? L1  = lfun(E,1,2)/2!
%48 = 0.75931650028842677023019260789472201908
```

Rational points

```
? E=ellinit([0,1,1,-7,5]);
? ellratpoints(E,100)
%2 = [[-1,3],[-1,-4],[1,0],[1,-1],[3,4],[3,-5],[5/4
%     [-47/16,161/64],[-47/16,-225/64],[85/49,225/3
? hyperellratpoints(x^6+x+1,100) \\ y^2 = x^6+x+1
%3 = [[-1,1],[-1,-1],[0,1],[0,-1],
%     [19/20,13109/8000],[19/20,-13109/8000]]
? (19/20)^6+(19/20)+1-(13109/8000)^2
%4 = 0
```

Heegner points

If E is of analytic rank 1, `ellheegner` return a non-torsion point on the curve.

```
? E = ellinit([-157^2,0]);
? lfunorderzero(E)
%5 = 1
? P = ellheegner(E)
%6 = [69648970982596494254458225/16613623166818526
%     538962435089604615078004307258785218335/67716
```

Isogenies

If E is a rational elliptic curve, `ellisomat(E)` computes representatives of the isomorphism classes of elliptic curves Q-isogenous to E.

```
? E=ellinit([0,1,1,-7,5]);
? lfunorderzero(E)
%2 = 1
? P = ellheegner(E)
%3 = [3,4]
? ellisoncurve(E,P)
%4 = 1
? [L,M]=ellisomat(E);
```

Isogenies

```
? M \\ isogeny matrix
%6 = [1,3,9;3,1,3;9,3,1]
? [e2,iso2,isod2]=L[2]
%7 = [[38/3,4103/108],
%     [x^3-5/3*x^2-11/3*x+16/3,
%      (y+1/2)*x^3+(-3*y-3/2)*x^2+(7*y+7/2)*x+(-7*y-
%      x-1],
%     [1/9*x^3+5/9*x^2+340/27*x+3527/243,
%      (1/27*y-1/2)*x^3+(4/9*y-6)*x^2+(-220/81*y-24)
%      x+4]]
```

Isogenies

```
? E2 = ellinit(e2);
? P2 = ellisogenyapply(iso2,P)
%10 = [19/12,63/8]
? ellisoncurve(E2,P2)
%11 = 1
? ellheight(E2,P2)/ellheight(E,P)
%12 = 3.000000000000000000000000000000000000
? Q = ellisogenyapply(isod2,P2)
%13 = [20901/17956,-759469/2406104]
? ellmul(E,P,3)
%14 = [20901/17956,-759469/2406104]
```

Cremona table and labels

(This require the package elldata)

```
? E=ellinit("11a1");
? ellglobalred(E)[1]
%2 = 11
? E=ellinit([3,4]);
? ellidentify(E)
%4 = [["1440i1",[0,0,0,3,4],[[0,2]]],[1,0,0,0]]
? ellconvertname("1440i1")
%5 = [1440,8,1]
? ellsearch(27)
%6 = [["27a1",[0,0,1,0,-7],[]],["27a2",[0,0,1,-270,
? forell(E,1,50,print(E))
```

Elliptic curves over number fields

We define the elliptic curve $y^2 + xy + \phi y = x^3 + (\phi + 1)x^2 + \phi x$ over the field $\mathbb{Q}(\sqrt{5})$ where $\phi = \frac{1+\sqrt{5}}{2}$. We set $a = \sqrt{5}$, $nf = \mathbb{Q}(\sqrt{5})$.

```
? nf  = bnfinit(a^2-5);
? phi = (1+a)/2;
? E   = ellinit([1,phi+1,phi,phi,0],nf);
? E.j
%4 = Mod(-53104/31*a-1649/31,a^2-5)
? E.disc
%5 = Mod(-8*a+17,a^2-5)
? N   = ellglobalred(E)[1]
%6 = [31,13;0,1]
? tor = elltors(E) \\ Z/8Z
%7 = [8,[8],[[-1,Mod(-1/2*a+1/2,a^2-5)]]]
```

Elliptic curves over number fields

We compute the reduction of the curve by the primes above 31.

```
? [pr1, pr2] = idealprimedec(nf,31);
? elllocalred(E,pr1) \\ multiplicative reduction
%9 = [1,5,[1,0,0,0],1]
? ellap(E,pr1) \\ -1: non-split
%10 = -1
? elllocalred(E,pr2) \\ good reduction
%11 = [0,0,[1,0,0,0],1]
? E2 = ellinit(E, pr2); \\ reduction of E mod pr2
? E2.j
%13 = Mod(13,31)
? ellap(E2)
%14 = 8
? ellgroup(E2) \\ Z/24Z
%15 = [24]
```

Elliptic curves over number fields

We check the BSD conjecture for E.

```
? om = E.omega
%16 = [[3.05217315,-2.39884477*I],
%       [8.43805989,4.21902994-1.57216679*I]]
? per = om[1][1]*om[2][1];
? tam = elltamagawa(E)
%18 = 2
? bsd = (per*tam) / (tor[1]^2*sqrt(abs(nf.disc)))
%19 = 0.35992895949803944944002575466348575048
? ellbsd(E)
%20 = 0.35992895949803944944002575466348575048
? L1  = lfun(E,1)
%21 = 0.35992895949803944944002575466348575048
```

Isogenies

```
? [L,M] = ellisomat(E);
? M
%23 = [1,2,4,4,8,8;
%      2,1,2,2,4,4;
%      4,2,1,4,8,8;
%      4,2,4,1,2,2;
%      8,4,8,2,1,4;
%      8,4,8,2,4,1]
```