Algebraic curves in PARI/GP, with an application to integrating algebraic functions

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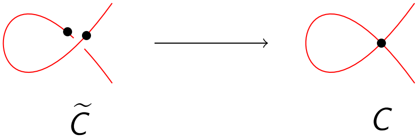
A package for plane algebraic curves

Plane algebraic curves in PARI/GP

Package to handle plane algebraic curves $C : F(x, y) = 0$.
Actually computes the desingularisation $\widetilde{C} \to C$ of (the projective closure of) C.



Main idea: represent "difficult" points of $\widetilde{C}$ by formal parametrisations $x(t), y(t)$.

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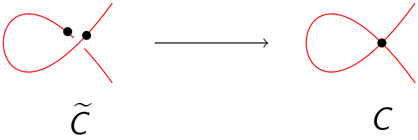


Main idea: represent "difficult" points of $\widetilde{C}$ by formal parametrisations $x(t), y(t)$.

Supported ground fields:

- Finite fields (but cannot handle some small characteristics),
- Fields of characteristic 0 (as long as PARI can factor polynomials over them...)

Example: Creation, divisors, Riemann-Roch

```
C=crvinit(x^11+y^7-2*x*y^5,t,a);
crvprint(C)

P=[1,1]
D=[P,-2;2,6;1,1]
crvdivprint(C,D);

L=crvRR(C,D)
crvfndiv(C,L[1],1);
crvfndiv(C,L[2],1);
```

Example: Rational curves

```
f=x^5+y^7+Mod(b,b^2-2)*x^3*y^3;
C=crvinit(f,t,a);
crvprint(C)

[T,param]=crvrat(C,1,3)

lift(param)
substvec(f,[x,y],param)

lift(T)
crvfndiv(C,T,1);

crvrat(C)
```

Example: Hyperelliptic / elliptic curves

```
C=crvinit(x^5+y^6+x^3*y,t,a);
crvprint(C)
crvishyperell(C)
crvhyperell(C)
```

```
C1=crvinit(x^5+y^7+x^3*y^4,t,a);
crvprint(C1)
crvell(C1,[1,-1,0])
```

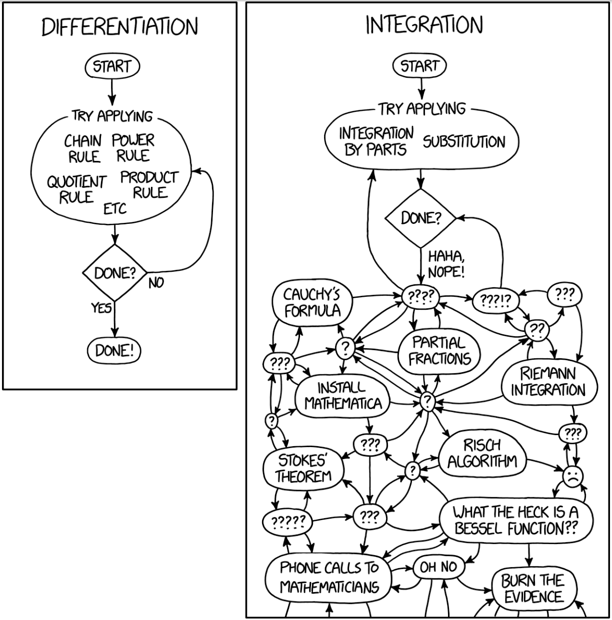
Other functionalities

- Over finite fields:
  Point counting, Zeta functions, group structure and word problem in Pic(C) (analogs of bnfinit and bnfisprincipal).

- Over number fields:
  Division polynomials, Galois representations.
  Bounding torsion of Pic(C), checking if divisors are torsion in Pic(C).

Symbolic integration of algebraic functions

Integration vs. differentiation



(Non-)elementary integrals

Complicated integrals often cannot be solved, e.g.

$$\int e^{-x^2}\,\mathrm{d}x \qquad \text{or} \qquad \int \frac{\mathrm{d}x}{\sqrt{x^3+1}}.$$

But then what about $\displaystyle\int \frac{\mathrm{d}x}{x} = \log x$?

Differential algebra

A differential field is a set of functions which is closed under $+, -, \times, \div, '$.
Example: $\mathbb{Q}(x)$.

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- y is logarithmic over $\mathcal{F}$ if $y' = f'/f$ for some $f \in \mathcal{F}$.
- y is exponential over $\mathcal{F}$ if $y'/y = f'$ for some $f \in \mathcal{F}$.

We then say that $\mathcal{F}(y)$ is a logarithmic / exponential extension of $\mathcal{F}$.

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An extension of $\mathcal{F}$ is elementary if it can be obtained from $\mathcal{F}$ as a finite succession of logarithmic / exponential / algebraic extensions.

An integral $\int f$ is elementary over $\mathcal{F}$ if there exists an elementary extension of $\mathcal{F}$ which contains a function F such that $F' = f$.

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Example

- $$\int \frac{\mathrm{d}x}{x^2+1} = \arctan x = \frac{1}{2\sqrt{-1}} \log\left(\frac{1+\sqrt{-1}x}{1-\sqrt{-1}x}\right)$$
  is elementary over $\mathbb{Q}(x)$.
- More generally, $f(x) \in \mathbb{Q}(x) \Longrightarrow \displaystyle\int f(x)\,\mathrm{d}x$ is elementary.
- OTOH, $\displaystyle\int e^{-x^2}\,\mathrm{d}x$ is not elementary over $\mathbb{C}(x)$.

Integrating algebraic functions

Let $f(x, y)$ be an algebraic function.
This means $f \in \mathcal{F}$ where $\mathcal{F} = \mathbb{Q}(C) = \mathbb{Q}(x)[y]/\big(F(x, y)\big)$ is the function field of a curve $C : F(x, y) = 0$.

Is $\displaystyle\int f(x, y)\, \mathrm{d}x$ elementary over $\mathcal{F}$? ($\Leftrightarrow$ over $\mathbb{Q}(x)$?)

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Is $\displaystyle\int f(x, y)\,\mathrm{d}x$ elementary over $\mathcal{F}$? ($\Leftrightarrow$ over $\mathbb{Q}(x)$?)

Usually not!

Example

$$\int \frac{x\,\mathrm{d}x}{\sqrt{x^4 + 10x^2 - 96x - m}}$$

is not elementary for most values of $m \in \mathbb{Q}$... but

$$\int \frac{x\,\mathrm{d}x}{\sqrt{x^4 + 10x^2 - 96x - 71}}$$

is elementary!

Liouville's criterion

Definition

Let $\mathcal{F}$ be a differential field. A Liouville sum over $\mathcal{F}$ is an expression of the form

$$\mathrm{d}g_0 + \sum_{i=1}^{m} c_i \frac{\mathrm{d}g_i}{g_i}$$

where $g_0, g_1, \cdots, g_m \in \mathcal{F}$ and $c_1, \cdots, c_m$ are constants.

Theorem (Liouville)

Let $\mathcal{F}$ be a differential field of characteristic 0, and let $f \in \mathcal{F}$.

$$\int f \text{ is elementary over } \mathcal{F} \iff f\,\mathrm{d}x \text{ is a Liouville sum over } \mathcal{F}.$$

Minimal Liouville sums

Let $\mathrm{d}g_0 + \sum_{i=1}^{m} c_i \frac{\mathrm{d}g_i}{g_i}$ be a Liouville sum over $\mathcal{F}$.

Pick a $\mathbb{Z}$-basis $e_1, \cdots, e_d$ of the $\mathbb{Z}$-span of $c_1, \cdots, c_m$

$$\rightsquigarrow c_i = \sum_{j=1}^{d} \lambda_{i,j} e_j, \quad \lambda_{i,j} \in \mathbb{Z}.$$

Then

$$\sum_{i=1}^{m} c_i \frac{\mathrm{d}g_i}{g_i} = \sum_{j=1}^{d} e_j \frac{\mathrm{d}G_j}{G_j}, \quad \text{where } G_j = \prod_{i=1}^{m} g_i^{\lambda_{i,j}} \in \mathcal{F}.$$

$\rightsquigarrow$ WLOG, we will assume m minimal, meaning that the c_i are $\mathbb{Q}$-linearly independent.

Controlling poles

Let $\mathcal{F} = \mathbb{Q}(C)$ function field of $C : F(x, y) = 0$, let $g \in \mathcal{F}$, and let $P \in C$.

- If g has a pole of order $n \geqslant 1$ at P, then $\mathrm{d}g$ has a pole of order $n + 1 \geqslant 2$ at P.
- If $\operatorname{ord}_P(g) = n \neq 0$, then $\dfrac{\mathrm{d}g}{g}$ has a <u>simple</u> pole at P with residue n.

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Consequence: Let $\omega = f(x)\,\mathrm{d}x$ be a meromorphic differential on a curve C.

If $\omega = \mathrm{d}g_0 + \displaystyle\sum_{i=1}^{m} c_i \frac{\mathrm{d}g_i}{g_i}$ is a Liouville sum,

- If all the poles of ω are simple, then g_0 has no poles, so $\mathrm{d}g_0 = 0$.
- Take m minimal; then the c_i form a $\mathbb{Q}$-basis of the $\mathbb{Q}$-span of the residues of $\displaystyle\sum_{i=1}^{m} c_i \frac{\mathrm{d}g_i}{g_i}$.
- In particular, if ω has no poles, then $\omega = 0$.

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Example

$\int \frac{\mathrm{d}x}{\sqrt{x^3+1}}$ is not elementary, because $\omega = \frac{\mathrm{d}x}{y}$ has no poles on $C : y^2 = x^3 + 1$.

Reduction to simple poles

Let $f \in \mathcal{F} = \mathbb{Q}(C)$, and let

$$(f\,\mathrm{d}x)_\infty = \sum_{k=1}^{m} n_k P_k$$

be the divisor of poles of $f\,\mathrm{d}x$.

$$f\,\mathrm{d}x = \mathrm{d}g_0 + \sum_{i=1}^{m} c_i \frac{\mathrm{d}g_i}{g_i} \Rightarrow g_0 \in \mathcal{L}(D), \text{ where } D = \sum_{k=1}^{m} (n_k - 1) P_k.$$

$\rightsquigarrow$ Look for $g_0 \in \mathcal{L}(D)$ such that $f\,\mathrm{d}x - \mathrm{d}g_0$ only has simple poles.

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$\rightsquigarrow$ Look for $g_0 \in \mathcal{L}(D)$ such that $f\,\mathrm{d}x - \mathrm{d}g_0$ only has simple poles.

If none exists, $\int f\,\mathrm{d}x$ is not elementary.

Otherwise, $\mathrm{d}g_0$ is unique, and $\displaystyle\int f\,\mathrm{d}x = g_0 + \int \omega_1$

where $\omega_1 = f\,\mathrm{d}x - \mathrm{d}g_0$ only has simple poles.

Torsion divisors

Let ω_1 have simple poles $P_1, \cdots, P_m \in C$ with residues $\rho_1, \cdots, \rho_m \in \overline{\mathbb{Q}}$.
Let $K = \mathbb{Q}(\rho_1, \cdots, \rho_m) \supseteq V = \mathbb{Q}$-span of $\rho_1, \cdots, \rho_m$.
Let $e_1, \cdots, e_d$ be a $\mathbb{Q}$-basis of V such that $\rho_k = \sum_{j=1}^d r_{k,j} e_j$ for some $r_{k,j} \in \mathbb{Z}$. Let $D_j = \sum_{k=1}^m r_{k,j} P_k \in \operatorname{Div}^0(C_K)$.

Claim: $\int \omega_1$ elementary $\Longrightarrow D_j$ is torsion in $\operatorname{Pic}^0(C_K)$ for all j.

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Indeed, suppose $\omega_1 = \sum_{i=1}^{n} c_i \frac{\mathrm{d}g_i}{g_i}$.

WLOG $d = n$ and the c_i form another $\mathbb{Q}$-basis of V, say

$e_j = \sum_{i=1}^{d} \frac{p_{i,j}}{q} c_i$ for some $p_{i,j}, q \in \mathbb{Z}$.

As $\sum_{j=1}^{d} r_{k,j} e_j = \rho_k = \sum_{i=1}^{d} c_i \operatorname{ord}_{P_k}(g_i)$,

$$\sum_{j=1}^{d} e_j D_j = \sum_{j=1}^{d} e_j \sum_{k=1}^{m} r_{k,j} P_k = \sum_{k=1}^{m} \rho_k P_k$$

$$= \sum_{k=1}^{m} \sum_{i=1}^{d} c_i \operatorname{ord}_{P_k}(g_i) = \sum_{i=1}^{d} c_i (g_i) = \sum_{j=1}^{d} e_j \sum_{i=1}^{d} \frac{p_{i,j}}{q} (g_i).$$

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Let $e_1, \cdots, e_d$ be a $\mathbb{Q}$-basis of V such that $\rho_k = \sum_{j=1}^d r_{k,j} e_j$ for some $r_{k,j} \in \mathbb{Z}$. Let $D_j = \sum_{k=1}^m r_{k,j} P_k \in \operatorname{Div}^0(C_K)$.

If some D_j is not torsion, then $\int \omega_1$ is not elementary.

Otherwise find $(g_j) = q_j D_j$ for $g_j \in K(C), q_j \in \mathbb{N}$. Then

$$\eta = \sum_{j=1}^d \frac{e_j}{q_j} \frac{\mathrm{d}g_j}{g_j} \quad \text{satisfies} \quad \operatorname{Res}_{P_k} \eta = \sum_{j=1}^d \frac{e_j}{q_j} q_j r_{k,j} = \rho_k = \operatorname{Res}_{P_k} \omega_1,$$

so $\omega_0 = \omega_1 - \eta$ has no poles, and

$$\int \omega_1 = \sum_{j=1}^d \frac{e_j}{q_j} \log(g_j) + \int \omega_0$$

is elementary $\iff \omega_0 = 0$.

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is elementary $\iff \omega_0 = 0$.

Remark

By Mordell-Weil, $\operatorname{Pic}^0(C_K) \simeq T \times \mathbb{Z}^r$.

f complicated $\implies K$ big $\implies r$ big.

A 31-year-old example

Groups

Conversations

Search conversations within sci. ...

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Dismiss Learn more

A Christmas present for your favorite CAS

274 views

Henri Cohen

to

Dec 21, 1993, 12:00:13 PM

Looking in my old files, I found the following INDEFINITE integral
(maple notation)

int(x/sqrt(x^4+10*x^2-96*x-71),x);

Of course this is an elliptic integral. However, it happens that this special integral can be computed explicitly. Questions:

1) Can any CAS compute this (not leaving the result with elliptic functions of course)? You are allowed to load any standard library you like.

2) Can YOU compute this?

3) Find other non-trivial examples.

Note: the experts in the field will know that there is a beautiful and rich theory behind this kind of computable elliptic integrals. In particular, relations with points of finite order on elliptic curves, and periodic continued fraction expansions. This can be considered as a (admittedly obscure) hint for non-experts.

A 31-year-old example

Consider $\displaystyle\int \frac{x\,\mathrm{d}x}{\sqrt{x^4+10x^2-96x-71}}$.

We introduce $C : y^2 = x^4+10x^2-96x-71$, and $\omega = \dfrac{x\,\mathrm{d}x}{y}$.

C has two points at infinity, ∞_+ and ∞_-, and ω has poles at ∞_+ and ∞_- only, both simple and with residue -1 and $+1$.

The integral won't be elementary unless the divisor $-\infty_+ + \infty_-$ is torsion in $\mathrm{Pic}^0(C)$.
Luckily, it is 8-torsion, as
$g = x^8 + (y+20)x^6 - 128x^5 + (15y+54)x^4 - (80y+1408)x^3 + (27y+3124)x^2 - 528yx + 781y + 10001$
has divisor $(g) = -8\infty_+ + 8\infty_-$.
And even more luckily,

$$\int \frac{x\,\mathrm{d}x}{\sqrt{x^4+10x^2-96x-71}} = \frac{1}{8}\log(g).$$

Another example

$$\int \frac{\sqrt[3]{x^8-6}}{x}\,\mathrm{d}x$$
$$= \frac{3}{8}\sqrt[3]{x^8-6}$$
$$+\tfrac{1}{16}a^2 \log\left(\frac{\left(a^4-4a\right)x^8+\left(48\sqrt[3]{x^8-6}^2+\left(3a^5-12a^2\right)\sqrt[3]{x^8-6}+\left(-6a^4+72a\right)\right)}{x^8}\right)$$
$$+\left(\tfrac{1}{128}a^5+\tfrac{1}{32}a^2\right)\log\left(\frac{8ax^8+\left(48\sqrt[3]{x^8-6}^2+\left(-3a^5-12a^2\right)\sqrt[3]{x^8-6}+\left(-6a^4-72a\right)\right)}{x^8}\right)$$

where $a^6+48=0$.

This involves spotting that some divisors on the genus 7 curve $y^3 = x^8 - 6$ defined over $\mathbb{Q}(\sqrt[6]{-48})$ are 8-torsion.

Testing for torsion

- Let C curve over a number field K, and $T = \mathrm{Pic}^0(C)_{\text{tors}}$.
If $\mathfrak{p}$ is a prime of K above $p \in \mathbb{N}$ such that C has good reduction at $\mathfrak{p}$, then

Reduction mod $\mathfrak{p}$ is injective on the prime-to-p part of T.

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  $\boxed{\text{Reduction mod } \mathfrak{p} \text{ is injective on the prime-to-}p\text{ part of } T.}$

- Let $\overline{C}/\mathbb{F}_q$ have genus g. Then

$$Z(\overline{C}/\mathbb{F}_q, t) = \exp \sum_{d=1}^{+\infty} \frac{\#\overline{C}(\mathbb{F}_{q^d})}{d} t^d = \frac{L(t)}{(1-t)(1-qt)}$$

  where $L(t) \in \mathbb{Z}[t]$ determined by $\#\overline{C}(\mathbb{F}_{q^d})$ for $d \leqslant g$.

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Let $D \in \operatorname{Div}^0(C)$. If m is small, we compute $\mathcal{L}(dD)$ for $d \mid m$.

If m is large, we check the order of D in $\operatorname{Pic}^0(\overline{C}_{\mathfrak{p}_i})$ by using Makdisi models.

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```
C=crvinit(x^9-y^5+2*x^4*y^2,t,a);
crvprint(C);
crvboundtorsion(C)
crvdivistorsion(C,[2,1;3,-1])
crvfndiv(C,%[2],1);

C=crvinit(y^2-x^6-2*x^5+3*x^4-8*x^3+8*x-4,t,a);
crvprint(C);
crvboundtorsion(C)
crvdivistorsion(C,[1,1;2,-1])
```

An example with 91-torsion

Let $f(x) = x^8 - 2x^7 + 7x^6 - 6x^5 - x^4 + 10x^3 - 6x^2 + 1$.

Then $\displaystyle\int \frac{2x^3 + 22x^2 + 47x - 91}{x\sqrt{f(x)}}\,\mathrm{d}x$

$= \log\left(A(x)\sqrt{f(x)} + B(x)\right) - 91\log(x)$, where $A(x) =$

$2541597392873x^{87} - 50843222146612x^{86} + 503225277935158x^{85} - 3200657096642275x^{84} + 14214462728604033x^{83} - 44579238719215767x^{82} +$
$90673772383763063x^{81} - 66130213758033706x^{80} - 273013962842426459x^{79} + 1133193576266076957x^{78} - 1828008617851129838x^{77} - 1325040205279907 92x^{76} +$
$7070565814431437671x^{75} - 13820814098546580816x^{74} + 30575014165909714 47x^{73} + 35452028969548856825x^{72} - 62530951562265159025x^{71} -$
$23621968960057272 08x^{70} + 149015656444634579168x^{69} - 167038416607981325445x^{68} - 122694173188447754583x^{67} + 429854211757535766713x^{66} -$
$169097783352406328449x^{65} - 555714282810473603258x^{64} + 674362321557037184728x^{63} + 312058060938121586273x^{62} - 1092460331914324201172x^{61} +$
$270596774739557247583x^{60} + 1120954182135661195118x^{59} - 880939983432258469781x^{58} - 730812820491441338716x^{57} + 1190924815315016075703x^{56} +$
$170419784195319443610x^{55} - 1106709092024065627293x^{54} + 266886129712577113986x^{53} + 775632662462383198827x^{52} - 447168828060446122800x^{51} -$
$414122686014061544643x^{50} + 415264647807791401896x^{49} + 156832329655217616311x^{48} - 289726675815819589903x^{47} - 261716891038418045 45x^{46} +$
$164791091923265170230x^{45} - 175169896340583532 70x^{44} - 79259644357109747485x^{43} + 20976219234985836422x^{42} + 3293254885810151040 7x^{41} -$
$13416187404910977913x^{40} - 12006472749426198850x^{39} + 6554509942630071562x^{38} + 3896330393014647662x^{37} - 2667133429777231104x^{36} -$
$1144094547215340652x^{35} + 936921199572723790x^{34} + 310346663095096540x^{33} - 289283382597149122x^{32} - 79724891819739155x^{31} + 79204013977345574x^{30} +$
$19845813628882518x^{29} - 19273182417066081x^{28} - 4834954816358415x^{27} + 4150468193299659x^{26} + 1140609211647771x^{25} - 781155386478148x^{24} -$
$253519603406578x^{23} + 125209807355899x^{22} + 51311674993204x^{21} - 16187503455853x^{20} - 9131100534854x^{19} + 1456557718427x^{18} + 1374884510502x^{17} -$
$30584589801x^{16} - 166171016046x^{15} - 18181479207x^{14} + 14582435700x^{13} + 3910302361x^{12} - 670862648x^{11} - 432933295x^{10} - 27794898x^{9} + 24199247x^{8} +$
$6635509x^{7} + 89529x^{6} - 311768x^{5} - 83944x^{4} - 11733x^{3} - 982x^{2} - 47x - 1$

and $B(x) \underset{\text{horror}}{\sim} A(x)$.

This is related to a rational 91-torsion point in $\mathrm{Pic}^0\left(y^2 - f(x)\right)$.
(Curve found by Steffen Müller and Berno Reitsma)

Final examples

Let $-x^5 + yx + y^4 = 0$ (genus 5).

Then $\displaystyle\int \frac{x^3}{y}\,\mathrm{d}x = \frac{4y^3}{11x} + \frac{1}{11}\log\left(\frac{y^3}{x}\right)$.

This involves spotting that some divisor is 11-torsion.

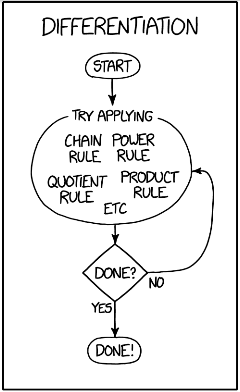
Our implementation takes 1 second; FriCAS takes 18 hours!

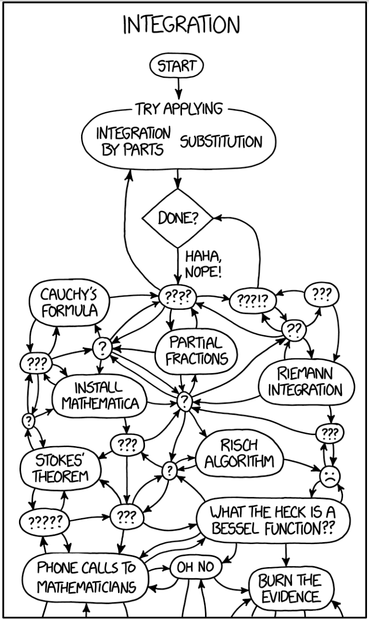
Same thing with

$$\int \frac{x^2 + 4y^3}{x^3}\,\mathrm{d}x = \frac{16y^3}{13x^2} + \frac{1}{13}\log\left(\frac{-x^{15} + 3yx^{10} - 3y^2x^5 + y^3}{x^{41}}\right)$$

where $-x^7 + yx^2 + y^4 = 0$ (genus 6, 13-torsion).

Conclusion


DIFFERENTIATION
START
TRY APPLYING
CHAIN RULE
POWER RULE
QUOTIENT RULE
PRODUCT RULE
ETC
DONE?
NO
YES
DONE!


INTEGRATION
START
TRY APPLYING
INTEGRATION BY PARTS
SUBSTITUTION
DONE?
HAHA, NOPE!
CAUCHY'S FORMULA
????
???!?
???
??
PARTIAL FRACTIONS
?
???
INSTALL MATHEMATICA
RIEMANN INTEGRATION
?
?
???
???
STOKES' THEOREM
RISCH ALGORITHM
?
?????
???
WHAT THE HECK IS A BESSEL FUNCTION??
PHONE CALLS TO MATHEMATICIANS
OH NO
BURN THE EVIDENCE

Questions?

Thank you!