Explicit computation of Satake parameters of automorphic representations

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Plan

1. Motivations
2. Lattices and automorphic forms for SO_n
3. Explicit computations

Automorphic representations

Notations: G semi-simple Lie group, $\widehat{G}$ Langlands dual, $\widehat{\mathfrak{g}}$ complex Lie algebra of $\widehat{G}$.
Denote by $\widehat{G}_{\mathrm{ss}}$ and $\widehat{\mathfrak{g}}_{\mathrm{ss}}$ the sets of semi-simple class of elements in $\widehat{G}$ and $\widehat{\mathfrak{g}}$.
An *automorphic representation* π is :

- a unitary representation of $G(\mathbb{R})\dots$
- $\dots$with a structure related to the Hecke algebra $\mathrm{H}(G)$.

Satake parametrisation

An automorphic representation π is determined by:

- its infinitesimal character: $c_\infty(\pi) \in \widehat{\mathfrak{g}}_{ss}$ (following Harish-Chandra);
- for every (unramified) prime p, its Satake parameter in p: $c_p(\pi) \in \widehat{G}_{ss}$ (following Satake and Langlands).

Automorphic representation as a generalisation of the classical modular forms for $\mathrm{SL}_2(\mathbb{Z})$:

- infinitesimal character is an analogous to the weight;
- Satake parameter in p is analogous to the p^{th} term in the q-expansion.

Examples from modular forms

Let $f = \sum a_n q^n$ be a modular eigenform of weight k. The associated automorphic representation π for PGL_2 is such that:

$$c_\infty(\pi) = \begin{pmatrix} \frac{k-1}{2} & 0 \\ 0 & -\frac{k-1}{2} \end{pmatrix} \quad \textit{and} \quad \chi(c_p(\pi))(X) = X^2 - p^{-\frac{k-1}{2}} a_p X + 1$$

Arthur's theory

Idea of the theory: given an automorphic representation π, there is a set of automorphic representations **for the linear groups**, whose Satake parameters give us the Satake parameters of π. This is part of **Langlands functoriality**.

Consequence: the Satake parameters of automorphic representations **for the linear groups** appear to be the **key elements** to understanding the automorphic representations for the classical groups.

Endoscopic representations

Idea: does a representation π for G comes from representation for smaller groups?
Arthur's theory associates to π a collection $\pi_1, \ldots, \pi_r$ of representations for $\mathrm{PGL}_{n_1}, \ldots, \mathrm{PGL}_{n_r}$. The representation π is said to be **endoscopic** if one of the n_i is smaller than the dimension of $\widehat{G}$.
Remarks:

- with the previous notations, if $r > 1$, then π is endoscopic;
- if π is a non-endosopic representations of SO_n, and π_1 the unique representation associated tu π by Arthur, then: $n_1 = n$ if n is even, and $n_1 = n - 1$ otherwise.

Some euclidean lattices

V dimension n euclidean space

$$\mathcal{L}_n := \{L \subset V,\ L \text{ even lattice}, \det(L) = 1 \textit{ ou } 2\}$$

Key point: the quadratic form induced from V is non-degenerate on L

$$\mathrm{X}_n := \mathrm{O}(V) \setminus \mathcal{L}_n$$

Proposition

We have $\mathcal{L}_n \neq \emptyset \Leftrightarrow n \equiv 0, \pm 1 \mod 8$.
More precisely, for such an n, if $L \in \mathcal{L}_n$:

$$\det(L) = 1 \Leftrightarrow n \equiv 0 \mod 8 \textit{ and } \det(L) = 2 \Leftrightarrow n \equiv \pm 1 \mod 8.$$

Some euclidean lattices

Definition

Given A an abelian group, two lattices $L_1, L_2 \in \mathcal{L}_n$ are said to be A-neighbours if they satisfy one of the following (equivalent) properties:

1. $L_1/(L_1 \cap L_2) \simeq A$;
2. $L_2/(L_1 \cap L_2) \simeq A$.

Definition

Given A as before, let T_A defined as:

$$\mathrm{T}_A(L) = \sum_{L' \ A-neighbour \ de \ L} L'.$$

d-neighbours

Particular case: when $A = \mathbb{Z}/d\mathbb{Z}$, we refer to “$d$-neighbours”.

Proposition (Parametrisation of d-neighbours)

Let $d \in \mathbb{N}^*$. The d-eighbours of a lattice $L \in \mathcal{L}_n$ are in a one-to-one correspondance with the isotropic simple $\mathbb{Z}/d\mathbb{Z}$-modules of L/dL of rank 1. If X is such a module, generated by $x \in L$ satisfying $x \cdot x \equiv 0 \mod 2d^2$, the d-neighbour of L corresponding to X is:

$$L' = X^{\perp} + \mathbb{Z}\frac{x}{d}$$

where $X^{\perp} = \{y \in L \mid x \cdot y \equiv 0 \mod d\}$.

Automorphic forms

We are interested in **cuspidal automorphic representations.**
In general: among the fonctions on $G(\mathbb{Q}) \setminus G(\mathbb{A})$, invariant by right translation of $G(\widehat{Z})$:

$$\{\text{cuspidal}\} \subset \{\text{discrete}\} \subset \{\text{square integrable}\}.$$

When $G = \mathrm{SO}_n$: everything that is automorphic is cuspidal. And we have the simple following definition:

Definition

Let (W, ρ) a finite dimension complex representation of $\mathrm{SO}_n(\mathbb{R}) \simeq \mathrm{SO}(V)$. The space of (cuspidal) automorphic representations for SO_n with weight W is the finite dimensional vector space:

$$\mathcal{M}_W(\mathrm{SO}_n) := \{f : \mathcal{L}_n \to W \mid \forall \gamma \in \mathrm{SO}_n(\mathbb{R}), f(\gamma \cdot L) = \rho(\gamma) \cdot f(L)\}.$$

Hecke operators on automorphic forms

Definition

The Hecke operators T_A, acting on the elements of $\mathcal{L}_n$, have a right-action on the space of automorphic forms:

$$\mathrm{T}_A(f)(L) = \sum_{L' \ A-\text{voisin de } L} f(L').$$

Theorem

Given n and W, there is an basis of $\mathcal{M}_W(\mathrm{SO}_n)$ of eigenvectors for all the operators T_A.

Automorphic forms and automorphic representations

Given $f \in \mathcal{M}_W(\mathrm{SO}_n)$, eigenform for all the operators T_A, there is a cuspidal automorphic representation π for SO_n, unramified at every place, associated to f. Moreover, π is related to f as follows:

- $c_\infty(\pi)$ only depends on W;
- for every prime p, $c_p(\pi)$ is determined by the eigenvalues of f for the T_A, where A is a group of order a power of p.

Theorem (Arthur, Taïbi)

The Langlands functoriality is satisfied when $G = \mathrm{SO}_n$. Moreover, the representations associated through Arthur's theory are cuspidal, algebraic, regular, autodual, unramified.

Some automorphic representations for the linear groups

The automorphic representations for PGL_n that we consider are:

- cuspidal, algebraic, regular;
- autodual;
- unramified.

"Well" known for $n \leq 5$:

- trivial for $n = 1$;
- classical modular forms for SL_2 for $n = 2, 3$;
- Siegel modular forms for Sp_4 for $n = 4, 5$.

Example: a representation of SO_7 comes from a representation of PGL_6 if, and only if, it is not endoscopic.

Trace of Hecke operators

Theorem (M.)

We have an explicit formula for the quantity $\mathrm{Trace}(\mathrm{T}_A|\mathcal{M}_W(\mathrm{SO}_n))$*, for any choice of* A*,* W *and* n*.*

The complexity of this formula comes from:

- the size of X_n;
- the number of orbits of A-neighbours of a lattice L under the action of $\mathrm{SO}(L)$;
- the size of the groups $\mathrm{SO}(L)$, for $L \in \mathcal{L}_n$.

More precisely: fixing A, for every class $\overline{L} \in \mathrm{X}_n$, we have a collection $\sigma_1, \ldots, \sigma_r \in \mathrm{SO}_n(\mathbb{Q})$, and we want to compute the multiset

$$\{\{\chi(\gamma \cdot \sigma_i) \mid \gamma \in \mathrm{SO}(L), i \in \{1, \ldots, r\}\}\}.$$

Explicit computations for only for $n = 7, 8, 9$ and A small.

Theorem (M.)

We know explicitely the quantities $\mathrm{Trace}(\mathrm{T}_A | \mathcal{M}_W(\mathrm{SO}_7))$ for any W, and A of one of the following forms:

- $(\mathbb{Z}/2\mathbb{Z})^i$, $i = 1, 2, 3$;
- $\mathbb{Z}/4\mathbb{Z}$;
- $\mathbb{Z}/q\mathbb{Z}$, $q \leq 67$ *power of a prime.*

The perfect situation

If $\dim(\mathcal{M}_W(\mathrm{SO}_n)) = 1$, then the trace computed is the eigenvalue we want.

Example for $n = 7$: when W is the representation of SO_7 of highest weight $(9, 5, 2)$ (following Chenevier–Renard), if π is associated to the unique element of $\mathcal{M}_W(\mathrm{SO}_n)$, then π is non-endoscopic and:

Theorem (M.)

The quantities $\tau(p) = p^{\frac{23}{2}} \cdot \mathrm{Trace}(c_p(\pi) \mid V_{\mathrm{St}})$ for $p \leq 67$ are given by:

p	2	3	5	7	...
$\tau(p)$	0	-304668	874314	452588136	...

The good situation

If there is only one non-endoscopic eigenform in $\mathcal{M}_W(\mathrm{SO}_n)$, we can substract the "endoscopic contribution" to the trace computed to get the eigenvalue.

Example for $n = 7$: when W is the representation of SO_7 of highest weight $(9,6,3)$, $\dim(\mathcal{M}_W(\mathrm{SO}_n)) = 2$ (following Chenevier–Renard), and if π_1, π_2 are the associated representation (with π_1 non endoscopic), then the Satake parameters for π_2 can be expressed by the unique parabolic modular form of weight 16 for SL_2, and the unique Siegel modular form of weight $\mathrm{Sym}^6\mathbb{C}^2 \otimes \det^{10}$ of genus 2 for Sp_4.

Theorem (M.)

The quantities $\tau(p) = p^{\frac{23}{2}} \cdot \mathrm{Trace}(c_p(\pi_1) \mid V_{\mathrm{St}})$ for $p \leq 67$ are given by:

p	2	3	5	7	...
$\tau(p)$	-720	425412	-124558326	-3040958424	...