Products of Eisenstein series

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Notations. For any integer $N \geq 1$, we define the congruence subgroups

$$\Gamma(N) = \left\{ g \in \mathrm{SL}_2(\mathbf{Z}) : g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

$$\Gamma_1(N) = \left\{ g \in \mathrm{SL}_2(\mathbf{Z}) : g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

$$\Gamma_0(N) = \left\{ g \in \mathrm{SL}_2(\mathbf{Z}) : g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}.$$

Note that $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N)$ and that $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbf{Z}/N\mathbf{Z})^\times$.

For any weight $k \geq 0$ and any congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$, the space $\mathcal{M}_k(\Gamma)$ of modular forms of weight k and level Γ decomposes into a direct sum

$$\mathcal{M}_k(\Gamma) = \mathcal{S}_k(\Gamma) \oplus \mathcal{E}_k(\Gamma).$$

The space of Eisenstein series $\mathcal{E}_k(\Gamma)$ is easy to describe (explicit bases are known), while the space of cusp forms $\mathcal{S}_k(\Gamma)$ is more mysterious.

A question

Let f be a modular form of weight $k \geq 2$ and some level.

Can we write f as a linear combination of products of two Eisenstein series of lower weights?

Potential applications.

1. Compute a large number of Fourier coefficients of f.
2. Compute the Fourier expansion of f at arbitrary cusps.
3. Compute the action of Atkin-Lehner operators on f.
4. Compute the automorphic representation associated to a newform f.

More precisely: let $k \geq 2$ and let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$. Multiplying Eisenstein series gives a linear map

$$\mu_{k,\Gamma} : \bigoplus_{k_1+k_2=k} \mathcal{E}_{k_1}(\Gamma) \otimes \mathcal{E}_{k_2}(\Gamma) \to \mathcal{M}_k(\Gamma).$$

What is the image of $\mu_{k,\Gamma}$?

- It is surjective for $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ (Kohnen–Zagier, 1977).
- It is surjective for $k \geq 3$ and $\Gamma = \Gamma_1(N)$ (Borisov–Gunnells, ~2000).
- It is surjective for $k = 2$ and $\Gamma = \Gamma(N)$ (Khuri-Makdisi, 2011).
- Under technical assumptions on N, every cusp form of weight $k \geq 4$ on $\Gamma_0(N)$ can be written using products of explicit Eisenstein series on $\Gamma_1(N)$ (Dickson–Neururer, 2016).

Remark. The map $\mu_{k,\Gamma}$ is not surjective in general: for $k = 2$ and $\Gamma = \Gamma_0(N)$, we have $\mathcal{E}_1(\Gamma_0(N)) = \{0\}$ so that the image of $\mu_{2,\Gamma_0(N)}$ is $\mathcal{E}_2(\Gamma_0(N))$.

Let us start with the definition of some Eisenstein series.

Definition

For a weight $k \geq 1$ and $(a,b) \in (\mathbf{Z}/N\mathbf{Z})^2$, we define

$$F_{a,b}^{(k)}(\tau) = c_{a,b}^{(k)} + \sum_{\substack{m,n \geq 1 \\ n \equiv a(N)}} \zeta_N^{bm} n^{k-1} q_N^{mn} + (-1)^k \sum_{\substack{m,n \geq 1 \\ n \equiv -a(N)}} \zeta_N^{-bm} n^{k-1} q_N^{mn},$$

where $c_{a,b}^{(k)} \in \mathbf{C}$ is an explicit constant, $\zeta_N = e^{2\pi i/N}$ and $q_N = e^{2\pi i \tau/N}$.

These Eisenstein series are well-behaved with respect to the action of $\mathrm{SL}_2(\mathbf{Z})$:

Proposition

For any $g \in \mathrm{SL}_2(\mathbf{Z})$, we have $F_{a,b}^{(k)}|_k g = F_{(a,b)\cdot g}^{(k)}$ except in the case $k = 2$ and $(a,b) = (0,0)$.

Definition

$$F_{a,b}^{(k)}(\tau) = c_{a,b}^{(k)} + \sum_{\substack{m,n \geq 1 \\ n \equiv a(N)}} \zeta_N^{bm} n^{k-1} q_N^{mn} + (-1)^k \sum_{\substack{m,n \geq 1 \\ n \equiv -a(N)}} \zeta_N^{-bm} n^{k-1} q_N^{mn},$$

In particular, we get the following propositions.

Proposition

The series $F_{a,b}^{(k)}$ is a modular form of weight k on $\Gamma(N)$, except in the case $k = 2$ and $(a, b) = (0, 0)$ where it is only quasi-modular.
In fact, the series $\{F_{a,b}^{(k)}\}_{a,b \in \mathbf{Z}/N\mathbf{Z}}$ span the space $\mathcal{E}_k(\Gamma(N))$.

Proposition

The series $F_{0,b}^{(k)}$ belongs to $\mathcal{E}_k(\Gamma_1(N))$, except in the case $k = 2$ and $b = 0$.

The work of Borisov and Gunnells

Borisov and Gunnells introduced the notion of *toric modular form*. They come from toric geometry, but can also be defined as follows.

Definition

Let $\mathcal{T}_*(N)$ be the graded subalgebra of $\mathcal{M}_*(\Gamma_1(N))$ generated by the weight 1 Eisenstein series $\{F_{0,b}^{(1)}\}_{b \in \mathbf{Z}/N\mathbf{Z}}$.
A toric modular form of weight k and level N is an element of $\mathcal{T}_k(N)$.

In particular, a toric modular form of weight 2 is a linear combination of products $F_{0,b}^{(1)} F_{0,b'}^{(1)}$ with $b, b' \in \mathbf{Z}/N\mathbf{Z}$.

The toric modular forms enjoy many remarkable properties.

Theorem (Borisov–Gunnells)

Assume $N \geq 5$.

1. *The ring $\mathcal{T}_*(N)$ is stable under the Hecke operators.*
2. *For $k \geq 3$, the cuspidal part of $\mathcal{T}_k(N)$ is the whole space $\mathcal{S}_k(\Gamma_1(N))$.*
3. *For $k = 2$, the cuspidal part of $\mathcal{T}_2(N)$ is the span of the eigenforms f satisfying $L(f,1) \neq 0$.*

Moreover, consider the map $\mu : (\mathbf{Z}/N\mathbf{Z})^2 \to \mathcal{M}_2(\Gamma_1(N))$ defined by

$$\mu(u,v) = F_{0,u}^{(1)} F_{0,v}^{(1)}.$$

4. *We have $\mu(u,v) + \mu(v,-u) = 0$.*
5. *We have $\mu(u,v) + \mu(v,-u-v) + \mu(-u-v,u) \in \mathcal{E}_2(\Gamma_1(N))$.*

The last part of the theorem implies that μ factors through modular symbols:

$$\begin{array}{ccc} E_N & \xrightarrow{\quad\mu\quad} & \mathcal{M}_2(\Gamma_1(N))/\mathcal{E}_2(\Gamma_1(N)). \\ & \searrow^{\xi} \quad \nearrow^{\overline{\mu}} & \\ & M_N & \end{array}$$

Here

- $E_N = \{(u,v) \in (\mathbf{Z}/N\mathbf{Z})^2 : \gcd(u,v,N) = 1\}$,
- $M_N = H_1(X_1(N), \{\text{cusps}\}, \mathbf{Z})$,
- $\xi(u,v)$ is the unimodular symbol $\{g0, g\infty\}$ for any matrix $g \in \mathrm{SL}_2(\mathbf{Z})$ such that $g \equiv \left(\begin{smallmatrix} * & * \\ u & v \end{smallmatrix}\right) \pmod{N}$.

Theorem (Borisov-Gunnells)

The map $\overline{\mu}$ is Hecke-equivariant.

Back to our original problem

Let E be an elliptic curve of conductor N, and let $f_E \in \mathcal{S}_2(\Gamma_0(N))$ be the newform of weight 2 associated to E.
We want to express f_E as a linear combination of pairwise products of Eisenstein series of weight 1.
The results of Borisov and Gunnells suggest the following strategy:

1. Compute the (minus) modular symbol $x_E^- \in M_N$ associated to E. We have $x_E^- = \sum_{(u,v)\in E_N} a_{u,v}\xi(u,v)$ for some $a_{u,v} \in \mathbf{Z}$.
2. Compute $F = \mu(x_E^-) = \sum a_{u,v} F_{0,u}^{(1)} F_{0,v}^{(1)}$.
3. By Hecke equivariance, we have $F = \alpha f_E \mod \mathcal{E}_2(\Gamma_1(N))$ for some $\alpha \in \mathbf{Q}(\zeta_N)$.
4. The main theorem then implies that $\alpha \neq 0$ precisely when $L(E,1) \neq 0$.

Problem. Computing with modular symbols on $\Gamma_1(N)$ is very expensive!

Definition

The trace map $\mathrm{Tr} : \mathcal{M}_2(\Gamma_1(N)) \to \mathcal{M}_2(\Gamma_0(N))$ *is defined by*

$$\mathrm{Tr}(f) = \sum_{\delta \in (\mathbf{Z}/N)^\times} f|_2 \langle \delta \rangle$$

where $\langle \delta \rangle \in \Gamma_0(N)$ *is any matrix congruent to* $\left(\begin{smallmatrix} \delta^{-1} & * \\ 0 & \delta \end{smallmatrix}\right)$ *modulo* N*.*

So, let's compute the trace of $\mu(u,v) = F^{(1)}_{0,u} F^{(1)}_{0,v}$.

$$\begin{aligned}
\mathrm{Tr}(\mu(u,v)) &= \sum_{\delta \in (\mathbf{Z}/N)^\times} F^{(1)}_{0,\delta u} F^{(1)}_{0,\delta v} \\
&\sim \sum_{\delta \in (\mathbf{Z}/N)^\times} \sum_{m,n \geq 1} \zeta_N^{\delta u m} q^{mn} \sum_{m',n' \geq 1} \zeta_N^{\delta v m'} q^{m'n'} \\
&\sim \sum_{m,m' \geq 1} \left(\sum_{\delta \in (\mathbf{Z}/N)^\times} \zeta_N^{\delta(um+vm')} \right) \sum_{n,n' \geq 1} q^{mn+m'n'} \\
&\sim \sum_{m,m' \geq 1} \mathrm{tr}_{\mathbf{Q}(\zeta_N)/\mathbf{Q}}(\zeta_N^{um+vm'}) \sum_{n,n' \geq 1} q^{mn+m'n'}
\end{aligned}$$

An analogue of μ for $\Gamma_0(N)$

We define

$$\begin{aligned}\mu_0 : \mathbf{P}^1(\mathbf{Z}/N\mathbf{Z}) &\to \mathcal{M}_2(\Gamma_0(N)) \\ (u:v) &\mapsto \operatorname{Tr}\mu(u,v) = \operatorname{Tr}(F^{(1)}_{0,u} F^{(1)}_{0,v}).\end{aligned}$$

1. The map μ_0 satisfies the Manin relations modulo $\mathcal{E}_2(\Gamma_0(N))$.
2. So we have a commutative diagram

$$\begin{array}{ccc} \mathbf{P}^1(\mathbf{Z}/N\mathbf{Z}) & \xrightarrow{\quad\mu_0\quad} & \mathcal{M}_2(\Gamma_0(N)) \\ \downarrow \xi & & \downarrow \\ H_1(X_0(N), \{\text{cusps}\}, \mathbf{Z}) & \xrightarrow{\bar{\mu}_0} & \mathcal{M}_2(\Gamma_0(N))/\mathcal{E}_2(\Gamma_0(N)). \end{array}$$

3. The map $\bar{\mu}_0$ is Hecke-equivariant.

Computations for elliptic curves

Given an elliptic curve $E/\mathbf{Q}$ of conductor N, we compute:

1. The Hecke eigensymbol $x_E^- \in H_1(X_0(N), \{\mathrm{cusps}\}, \mathbf{Z})^-$.
2. The modular form $F = \mu_0(x_E^-)$.

Then we know that $F = \alpha_E f_E + G$ for some $\alpha_E \in \mathbf{Q}$ and $G \in \mathcal{E}_2(\Gamma_0(N))$.

Actually it seems to be the case that G belongs to the subspace $\mathcal{E}_2'(\Gamma_0(N))$ generated by $\{\mathrm{Tr}(F_{0,b}^{(2)})\}_{b \in \mathbf{Z}/N\mathbf{Z}, b \neq 0}$. In general $\mathcal{E}_2'$ is a proper subspace of $\mathcal{E}_2$.

We checked that for all elliptic curves of conductor $N \leq 10^3$ we have

$$\alpha_E \neq 0 \Leftrightarrow L(E,1) \neq 0$$

in accordance with Borisov–Gunnells.

Computations for elliptic curves

We have

$$F = \alpha_E f_E + \sum_{\substack{d|N \\ d \neq N}} \alpha_d \operatorname{Tr}(F_{0,d}^{(2)}) \qquad (\alpha_E, \alpha_d \in \mathbf{Q}). \tag{1}$$

How to compute α_E and α_d?

- Since f_E vanishes at all cusps, it is sufficient to evaluate F and $\operatorname{Tr}(F_{0,d}^{(2)})$ at the cusps to get the α_d's.
- Since $f_E = q + O(q^2)$, it is sufficient to compute the coefficient of q of both sides of (1) to get α_E.

Remark. It gives an algebraic method to decide whether $L(E,1)$ is zero or not.

Question. According to the data α_E is a rational number with small denominator. Is there a formula for α_E?

A linear algebra question

In order to speed up the computation of $F = \mu_0(x_E^-)$, we want to express x_E^- as a linear combination of *fewest possible* unimodular symbols $\{g0, g\infty\}$.

By Manin's theorem, we have a presentation of $H = H_1(X_0(N), \{\text{cusps}\}, \mathbf{Q})$:

- Generators: $\xi(u:v) = \{g_{u,v}0, g_{u,v}\infty\}$ for $(u:v) \in \mathbf{P}^1(\mathbf{Z}/N\mathbf{Z})$.
- Manin's relations: $\begin{cases} \xi(u:v) + \xi(v:-u) = 0, \\ \xi(u:v) + \xi(v:-u-v) + \xi(-u-v:u) = 0. \end{cases}$.

In order to get the minus part H^-, we set $\xi^-(u:v) := \frac{1}{2}(\xi(u:v) - \xi(-u:v))$. Then H^- is generated by the $\xi^-(u:v)$ together with the obvious relations.

What is the shortest expression of x_E^- in terms of Manin symbols $\xi^-(u:v)$?

What to do if $L(E, 1) = 0$?

Possible approaches:

1. Multiply f_E by a known Eisenstein series of weight 2 to get a modular form g of weight 4. Then apply Borisov-Gunnells' theory to g.
   Problem: g is a cusp form but not necessarily an eigenform.
2. Use Eisenstein series on $\Gamma(N)$ and Khuri-Makdisi's theorem.
   Note that the trace map $\mathcal{M}_2(\Gamma(N)) \to \mathcal{M}_2(\Gamma_1(N))$ is very easy to compute: it sends $\sum a_n q_N^n$ to $\sum a_{Nn} q^n$, so there is some hope to do something similar as above.

Thank you for your attention!