Bianchi group calculations in PARI/GP

Alexander D. Rahm

Lecturer at the National University of Ireland at Galway

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Information on the Bianchi groups that can be computed with the Bianchi.gp script in PARI/GP:

- Orbit space of the action of the Bianchi groups on hyperbolic space
- Symmetry-subdivided cell structure with stabilisers and identifications
- Equivariant K-homology
- Group homology
- Chen–Ruan orbifold cohomology

The Bianchi groups

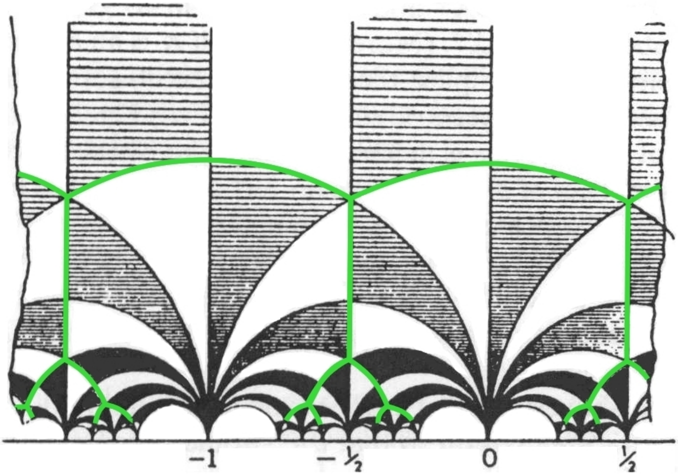
Definition

For m a positive square-free integer, let $\mathcal{O}_{-m}$ denote the ring of algebraic integers in the imaginary quadratic field extension $\mathbb{Q}[\sqrt{-m}]$ of the rational numbers.
The Bianchi groups *are the projective special linear groups $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$.*

Motivations

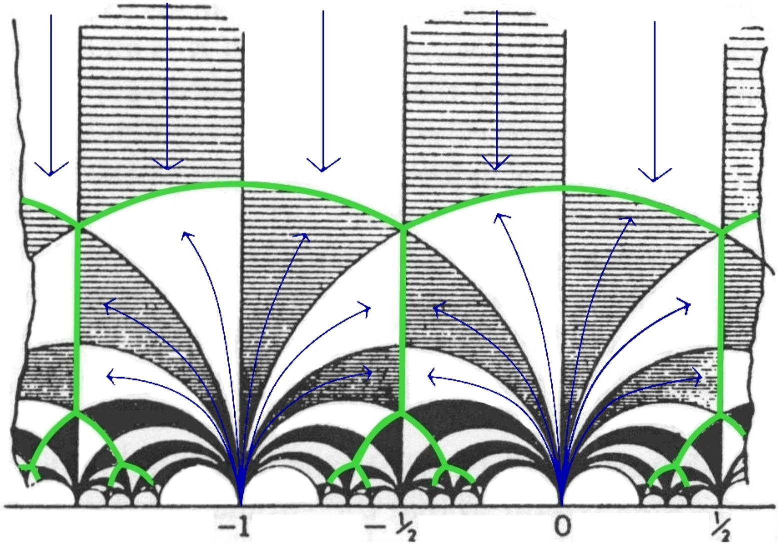
- Group theory
- Hyperbolic geometry
- Knot theory
- Automorphic forms
- Baum/Connes conjecture
- Algebraic K-theory
- Heat kernels
- Quantized orbifold cohomology

The upper-half space model acted on by $PSL_2(\mathbb{Z})$

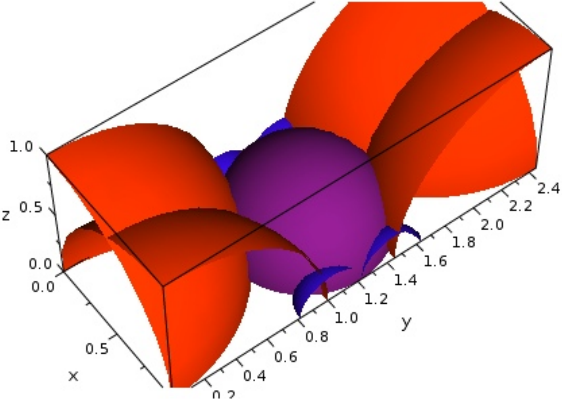


Underlying picture by Robert Fricke for Felix Klein's lecture notes, 1892

The $PSL_2(\mathbb{Z})$-equivariant retraction



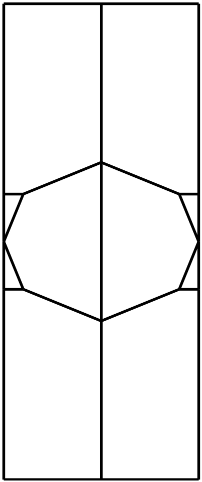
A fundamental domain for $\Gamma := \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-6}\,])$

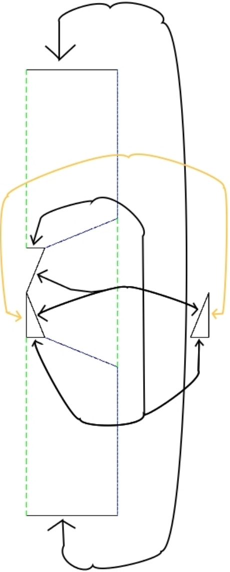


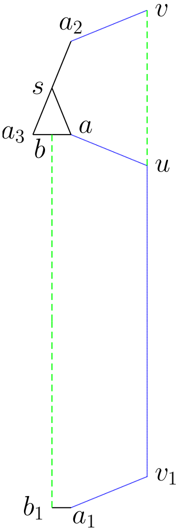
(Picture created by Mathias Fuchs, using coordinates from Bianchi.gp)

$$\begin{array}{ccccc} \mathrm{PSL}_2(\mathbb{Z}) & \hookrightarrow & \mathrm{PSL}_2(\mathbb{R}) & \circlearrowright & \mathcal{H}^2_{\mathbb{R}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{PSL}_2(\mathcal{O}_{-m}) & \hookrightarrow & \mathrm{PSL}_2(\mathbb{C}) & \circlearrowright & \mathcal{H}^3_{\mathbb{R}} \end{array}$$

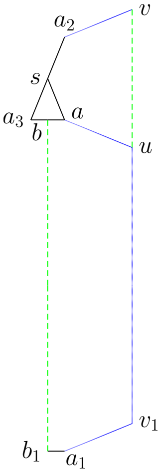
Cell structure for $\Gamma := \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-6}\,])$







Cell stabilisers in $\Gamma := \mathrm{PSL}_2(\mathbb{Z}[\omega])$, $\omega := \sqrt{-6}$



$$A := \pm\begin{pmatrix} & -1 \\ 1 & \end{pmatrix},$$

$$B := \pm\begin{pmatrix} -1-\omega & 2-\omega \\ 2 & 1+\omega \end{pmatrix},$$

$$R := \pm\begin{pmatrix} -\omega & 5-\omega \\ 1 & 1+\omega \end{pmatrix},$$

$$S := \pm\begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix},$$

$$V := \pm\begin{pmatrix} 1-\omega & 3 \\ 3 & 1+\omega \end{pmatrix},$$

$$W := \pm\begin{pmatrix} 7 & 3\omega \\ 2\omega & -5 \end{pmatrix}.$$

$$\Gamma_u = \left\langle B,\ S \mid B^2 = S^3 = (BS)^3 = 1 \right\rangle \cong \mathcal{A}_4,$$

$$\Gamma_v = \left\langle B,\ R \mid B^2 = R^3 = (BR)^3 = 1 \right\rangle \cong \mathcal{A}_4,$$

$$\Gamma_a = \left\langle SB \mid (SB)^3 = 1 \right\rangle \cong \mathbb{Z}/3\mathbb{Z},$$

$$\Gamma_b = \left\langle A \mid A^2 = 1 \right\rangle \cong \mathbb{Z}/2\mathbb{Z},$$

$$\Gamma_s = \langle V,\ W \mid VW = WV \rangle \cong \mathbb{Z}^2.$$

History of computations for the Bianchi groups

Luigi Bianchi (1856-1928)

- Fundamental domains in eleven cases by Bianchi, 1892
- Concept for an algorithm by Swan, 1971
- Fundamental polyhedra implementation by Riley, 1983
- Mendoza complex implementation by Vogtmann, 1985
- Fundamental domains by Cremona and students, 1984-2010
- Program for $\mathrm{GL}_2(\mathcal{O})$ by Yasaki, 2010 (Gunnells' algorithm)
- Recent SAGE package for Bianchi groups by Maite Aranes
- Recent MAGMA package for Kleinian groups by Aurel Page

Features of the Bianchi.gp script in PARI/GP

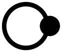
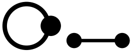
- Computation of the group cohomology, especially of the torsion Grunewald–Poincaré series
- Computation of the Bredon homology for operator K-theory
- Quantized orbifold cohomology computations
- Dimension computation for Bianchi modular forms spaces

Fritz Grunewald (1949-2010)

$$P^{\ell}(t) := \sum_{q = \mathrm{vcd}(\Gamma)+1}^{\infty} \dim_{\mathbb{F}_\ell} \mathrm{H}_q\left(\Gamma;\, \mathbb{Z}/\ell\right)\, t^q.$$

Some results in homological 3–torsion

Let $P_m^3(t) := \sum_{q=3}^{\infty} \dim_{\mathbb{F}_3} \mathrm{H}_q(\mathrm{PSL}_2(\mathcal{O}_{\mathbb{Q}[\sqrt{-m}]}); \mathbb{Z}/3) t^q.$

m specifying the Bianchi group	3–torsion subcomplex, homeomorphism type	$P_m^3(t)$
2, 5, 6, 10, 11, 15, 22, 29, 34, 35, 46, 51, 58, 87, 95, 115, 123, 155, 159, 187, 191, 235, 267		$\frac{-2t^3}{t-1}$
7, 19, 37, 43, 67, 139, 151, 163		$\frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$
13, 91, 403, 427		$2\left(\frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}\right)$
39		$\frac{-2t^3}{t-1} + \frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$

Group homology

Theorem (R., October 2011)

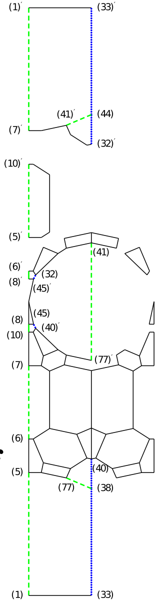
For all Bianchi groups with units $\{\pm 1\}$, the homology in all degrees above their virtual cohomological dimension is given by the following Poincaré series:

$$P_m^2(t) = \left(\lambda_4 - \frac{3\mu_2 - 2\mu_T}{2}\right) P_{\bigcirc\!\!\bullet}(t) + (\mu_2 - \mu_T) P^*_{\mathcal{D}_2}(t) + \mu_T P^*_{\mathcal{A}_4}(t)$$

$$\text{and } P_m^3(t) = \left(\lambda_6 - \frac{\mu_3}{2}\right) P_{\bigcirc\!\!\bullet}(t) + \frac{\mu_3}{2} P_{\bullet\!-\!\bullet}(t),$$

where $P_{\bigcirc\!\!\bullet}(t) := \frac{-2t^3}{t-1}$, $\quad P_{\bullet\!-\!\bullet}(t) := \frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$,

and the numbers	μ_2	μ_3	μ_T	λ_{2n}
count conjugacy classes of	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$	$\mathbb{Z}/n$



For a prime ℓ, consider the subcomplex of the cell complex consisting of the cells with elements of order ℓ in their stabiliser.
We call it the ℓ*–torsion subcomplex*.

Torsion subcomplex reduction

In the ℓ-torsion subcomplex, let σ be a cell of dimension $n-1$, lying in the boundary of precisely two n-cells τ_1 and τ_2, the latter cells representing two different orbits. Assume further that no higher-dimensional cells of the ℓ-torsion subcomplex touch σ.
Condition for merging cells : $\widehat{\mathrm{H}}^*(\Gamma_{\tau_1})_{(\ell)} \cong \widehat{\mathrm{H}}^*(\Gamma_{\tau_2})_{(\ell)} \cong \widehat{\mathrm{H}}^*(\Gamma_\sigma)_{(\ell)}$.

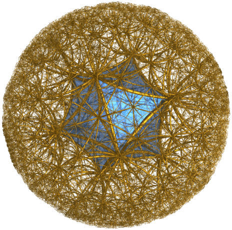
This can be obtained with $\Gamma_{\tau_1} \cong \Gamma_{\tau_2}$, Γ_σ being ℓ-normal and $\Gamma_{\tau_1} \cong \mathrm{N}_{\Gamma_\sigma}(\mathrm{center}(\mathrm{Sylow}_p(\Gamma_\sigma)))$.

Lemma

Let $\widetilde{X_\ell}$ be the Γ-complex obtained by orbit-wise merging two n-cells of the ℓ-torsion subcomplex X_ℓ satisfying the merging conditions. Then,

$$\widehat{\mathrm{H}}^*(\Gamma)_{(\ell)} \cong \widehat{\mathrm{H}}^*_\Gamma(\widetilde{X_\ell})_{(\ell)}.$$

Classical method	Systematic torsion subcomplex reduction
Calculation of the orbit space $\underline{\mathrm{E}}\Gamma/_\Gamma$	Calculation of the orbit spaces of the torsion subcomplexes



(image source: Claudio Rocchini)

Limited number of examples

General formulas for the torsion part
- of the homology of the groups $\mathrm{SL}_2(A)$ over any number ring A (j/w M. Wendt)
- of the homology of the Coxeter tetrahedral groups
- of quantised orbifold cohomology
- of equivariant K-homology (to appear soon)

The Bredon chain complex for equivariant K-homology

$$
\begin{array}{c}
0 \\
\downarrow \\
\bigoplus_{\sigma \in {}_\Gamma\backslash X^{(2)}} R_{\mathbb{C}}(\Gamma_\sigma) \\
\downarrow \Psi_2 \\
\bigoplus_{\sigma \in {}_\Gamma\backslash X^{(1)}} R_{\mathbb{C}}(\Gamma_\sigma) \\
\downarrow \Psi_1 \\
\bigoplus_{\sigma \in {}_\Gamma\backslash X^{(0)}} R_{\mathbb{C}}(\Gamma_\sigma) \\
\downarrow \\
0
\end{array}
$$

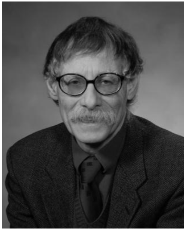
Theorem (R.)
Let $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$. Then, for $\mathcal{O}_{-m}$ principal, the equivariant K-homology of Γ has isomorphy types

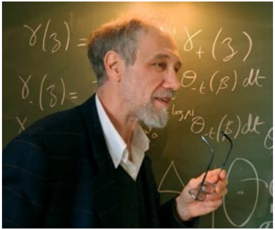
	$m = 1$	$m = 2$	$m = 3$	$m = 7$	$m = 11$	$m \in \{19, 43, 67, 163\}$
$K_0^\Gamma(\underline{E}\Gamma)$	$\mathbb{Z}^6$	$\mathbb{Z}^5 \oplus \mathbb{Z}/2$	$\mathbb{Z}^5 \oplus \mathbb{Z}/2$	$\mathbb{Z}^3$	$\mathbb{Z}^4 \oplus \mathbb{Z}/2$	$\mathbb{Z}^{\beta_2} \oplus \mathbb{Z}^3 \oplus \mathbb{Z}/2$
$K_1^\Gamma(\underline{E}\Gamma)$	$\mathbb{Z}$	$\mathbb{Z}^3$	0	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z} \oplus \mathbb{Z}^{\beta_1}$,

where the Betti numbers are

m	19	43	67	163
β_1	1	2	3	7
β_2	0	1	2	6.

The (analytical) assembly map





Paul Frank Baum

Alain Connes

$$\mu_i : K_i^G(\underline{E}G) \longrightarrow K_i(C_r^*(G)), \qquad i \in \mathbb{N} \cup \{0\}$$

The vector space structure of the Chen/Ruan quantized orbifold cohomology

Let Γ be a discrete group acting by diffeomorphisms on a manifold Y, with finite stabilisers.

Definition

Let $T \subset \Gamma$ be a set of representatives of the conjugacy classes of elements of finite order of Γ. Set

$$\mathrm{H}^*_{orb}(Y//\Gamma) := \bigoplus_{g \in T} \mathrm{H}^*\left(Y^g / C_\Gamma(g); \mathbb{Q}\right).$$

Dimensions of Chen/Ruan orbifold cohomology

Theorem (R.)

$$\mathrm{H}^d_{orb}\left(\mathcal{H}//PSL_2(\mathcal{O}_{-m})\right) \cong \begin{cases} \mathbb{Q}, & d = 0 \\ \mathbb{Q}^{\beta_1}, & d = 1 \\ \mathbb{Q}^{\beta_1 - 1 + \lambda_4 + 2\lambda_6 - \lambda_6^*}, & d = 2 \\ \mathbb{Q}^{\lambda_4 - \lambda_4^* + 2\lambda_6 - \lambda_6^*}, & d = 3 \\ 0 & \text{otherwise.} \end{cases}$$

The Eichler–Shimura–Harder isomorphism

Let $\Gamma := \mathrm{SL}_2(\mathcal{O}_{-m})$. **The space $\mathrm{H}^1_{\mathrm{cusp}}(\Gamma;\, E_{k,k})$ is isomorphic to the space of weight $k+2$ cuspidal automorphic forms of Γ over $\mathbb{Q}(\sqrt{-m})$.**

Discovered cases where there are *genuine* classes, where D is the discriminant of $\mathbb{Q}(\sqrt{-m})$.

$\|D\|$	**7**	**11**	**71**	**87**	**91**	**155**	**199**	**223**	**231**	**339**
k	12	10	1	2	6	4	1	0	4	1
dim	2	2	2	2	2	2	4	2	2	2

$\|D\|$	**344**	**407**	**408**	**408**	**408**	**415**	**435**	**435**	**435**	**435**
k	1	0	2	5	8	0	2	5	8	11
dim	2	2	2	2	2	2	2	2	2	2

$\|D\|$	**455**	**483**	**571**	**571**	**643**	**760**	**1003**	**1003**	**1051**	
k	0	1	0	1	0	2	0	1	0	
dim	2	2	2	2	2	2	2	2	2	

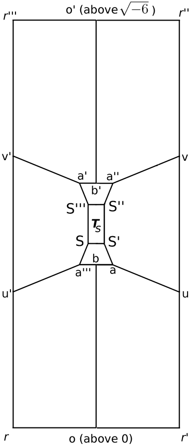
Let Γ be a finite index subgroup in a Bianchi group. Consider the Borel–Serre compactification ${}_{\Gamma}\backslash\widehat{\mathcal{H}}$ of the orbit space ${}_{\Gamma}\backslash\mathcal{H}$.
Its boundary $\partial({}_{\Gamma}\backslash\widehat{\mathcal{H}})$ consists of a 2-torus at each cusp s.
Consider the map α induced on homology when attaching the boundary $\partial({}_{\Gamma}\backslash\widehat{\mathcal{H}})$ into ${}_{\Gamma}\backslash\widehat{\mathcal{H}}$.

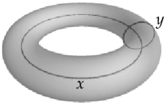
Theorem (Serre, 1970)

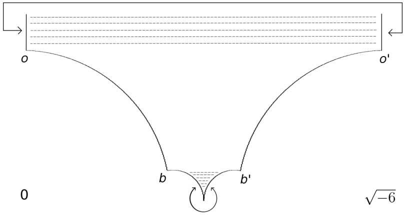
Suppose that the coefficient module M is equipped with a non-degenerate Γ-invariant $\mathbb{C}$-bilinear form. Then the rank of the map from $H^1(\Gamma;\, M)$ to the disjoint sum of the $H^1(\Gamma_s;\, M)$, induced by α, equals half of the rank of the disjoint sum of the $H^1(\Gamma_s;\, M)$.

Question (Serre 1970).
How can one determine the kernel of α (in degree 1) ?

R., "*On a question of Serre*",
Note aux CRAS présentée par J.-P. Serre







Thank you
for your attention!