A Brumer-Stark conjecture for Galois extensions

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K/k abelian extension of number fields

$G = \mathrm{Gal}(K/k)$, its Galois group

$S \subset \mathsf{Pl}(k)$ finite, containing $\mathsf{Pl}_\infty(k)$ and $\mathsf{Pl}_{\mathsf{ram}}(K/k)$

For $\chi \in \hat{G}$, $L_{K/k,S}(s,\chi) = \prod_{\mathfrak{p} \notin S} \left(1 - \chi(\sigma_\mathfrak{p})\mathcal{N}\mathfrak{p}^{-s}\right)^{-1}$ (Hecke L-function)

The Brumer-Stickelberger element

$$\theta_{K/k,S} := \sum_{\chi \in \hat{G}} L_{K/k,S}(0, \bar{\chi}) e_\chi$$

The Brumer-Stickelberger element is the only element of $\mathbb{C}[G]$ such that

$$\chi(\theta_{K/k,S}) = L_{K/k,S}(0, \bar{\chi}), \ \forall \chi \in \hat{G}$$

Let $v \in S$ and let $N_v = \sum_{\sigma \in D_v} \sigma$. Then

$$N_v \cdot \theta_{K/k,S} = 0$$

In particular, if there exists $v \in S$ totally split, then $\theta_{K/k,S} = 0$.

Let $\mathfrak{p}$ be a prime ideal of k not in S. Then

$$\theta_{K/k,S\cup\{\mathfrak{p}\}} = \theta_{K/k,S} \cdot (1 - \sigma_{\mathfrak{p}}^{-1})$$

For all $\xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, we have $\xi \cdot \theta_{K/k,S} \in \mathbb{Z}[G]$

The Brumer-Stark Conjecture $\mathbf{BS}(K/k,S)$

Let $w_K := |\mu_K|$. We have

$$w_K\,\theta_{K/k,S} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathsf{Cl}_K)$$

Furthermore, for any fractional ideal $\mathfrak{A}$ of K, there exists $\alpha \in K^\times$ with

- $\mathfrak{A}^{w_K\theta_{K/k,S}} = (\alpha)$
- $\alpha \in K^\circ$
- $K(\alpha^{1/w_K})/k$ is abelian

$\Bigg\}\ \mathbf{BS}(K/k,S;\mathfrak{A})$

Remarks.

- $K^\circ := \{x \in K^\times : |x|_w = 1,\ \forall w \in \mathsf{Pl}_\infty(k)\}$ (anti-units)
- If K not totally complex or k not totally real, then $\theta_{K/k,S} = 0$.

For $\mathfrak{A}$ be a fractional ideal of K, $\mathbf{BS}(K/k, S; \mathfrak{A})$ is equivalent to

1. There exists an extension L/K such that L/k is abelian and an anti-unit $\gamma \in L^\circ$ such that $(\mathfrak{A}\mathcal{O}_L)^{\theta_{K/k,S}} = \gamma\mathcal{O}_L$.
2. For almost all prime ideals $\mathfrak{p}$ of k, there exists $\alpha_\mathfrak{p} \in K^\circ$ such that $\mathfrak{A}^{(\sigma_\mathfrak{p} - \mathcal{N}(\mathfrak{p}))\theta_{K/k,S}} = \alpha_\mathfrak{p}\mathcal{O}_K$ and $\alpha_\mathfrak{p} \equiv 1 \pmod{^* \mathfrak{p}\mathcal{O}_K}$.
3. There exist a family $(a_i)_{i\in I}$ of element of $\mathbb{Z}[G]$ generating $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ and a family $(\alpha_i)_{i\in I} \subset K^\circ$ such that $\mathfrak{A}^{a_i\theta_{K/k,S}} = \alpha_i\mathcal{O}_K$ and ${\alpha_i}^{a_j} = {\alpha_j}^{a_i}$ for all $i, j \in I$.

The set $\{\mathfrak{A} : \mathbf{BS}(K/k, S; \mathfrak{A}) \text{ is true}\}$ is a group stable under the action of G that contains the principal ideals of K.

Assume $k \subset F \subset K$. Then $\mathbf{BS}(K/k, S) \implies \mathbf{BS}(F/k, S)$.

$\mathbf{BS}(K/k,S)$ is true in the following cases

- $k=\mathbb{Q}$. [Stickelberger theorem]
- K is principal. [Tate]
- K/k of degree 4 contained in K/k_0 Galois but not abelian of degree 8. [Tate]
- G is of exponent 2 ($+$ some technical conditions). [Sands]
- Many numerical cases with k of degree 2, 3 or 4. [Greither, R., Tangedal]

Base change [Hayes]. Let $K/k'/k$ abelian.

Then $\mathbf{BS}(K/k,S) \implies \mathbf{BS}(K/k',S')$ with $S' := \{v' \in \mathrm{Pl}(k') : v'_{|k} \in S\}$.

The local Brumer-Stark Conjecture $\mathbf{BS}^{(\ell)}(K/k,S)$

Let $w_{K,\ell}$ be the ℓ-part of w_K. We have

$$w_K\,\theta_{K/k,S} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathsf{Cl}_K\{\ell\})$$

Furthermore, for $\mathfrak{A}$ with $[\mathfrak{A}] \in \mathrm{Cl}_K\{\ell\}$, there exists $\alpha \in K^\times$ with

- $\mathfrak{A}^{w_K\theta_{K/k,S}} = (\alpha)$
- $\alpha \in K^\circ$
- $K(\alpha^{1/w_{K,\ell}})/k$ is abelian

We have
$$\mathbf{BS}(K/k,S) \iff \mathbf{BS}^{(\ell)}(K/k,S)\ \forall \ell$$

Proved in many cases of degree $2p$ [Greither, R., Tangedal; Smith].

$\mathbf{BS}^{(\ell)}(K/k,S)$ is proved for $\ell \neq 2$ by Popescu-Greither provided the adequate Iwasawa μ-invariant vanishes.

In order to generalize the Brumer-Stark conjecture to the Galois case, we need to generalize the following:

The Brumer-Stickelberger element: $\theta_{K/k,S} := \sum_{\chi \in \hat{G}} L_{K/k,S}(0, \bar{\chi}) e_\chi$

The Brumer part: $w_K \theta_{K/k,S} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathsf{Cl}_K)$

The Stark part: $\mathfrak{A}^{w_K \theta_{K/k,S}} = (\alpha)$ with $\alpha \in K^\circ$ and $K(\alpha^{1/w_K})/k$ abelian

(Another generalization in a different direction has been done by A. Nickel)

We assume from now on that K/k is a Galois extension with group G.

After Hayes, define

$$\theta_{K/k,S} := \sum_{\chi \in \hat{G}} L_{K/k,S}(0,\bar{\chi})\, e_\chi \in Z(\mathbb{C}[G])$$

where $\hat{G}$ is the set of irreducible characters of G and

$$e_\chi := \frac{\chi(1)}{|G|} \sum_{g \in G} \bar{\chi}(g)\, g$$

are the idempotents of $Z(\mathbb{C}[G])$.

Clearly, we recover the previous definition of $\theta_{K/k,S}$ when K/k is abelian.

Recall the properties of the Brumer-Stark element in the abelian case.

The Brumer-Stickelberger element is the only element of $\mathbb{C}[G]$ such that

$$\chi(\theta_{K/k,S}) = L_{K/k,S}(0, \bar{\chi}), \ \forall \chi \in \hat{G}$$

Let $v \in S$ and let $N_v = \sum_{\sigma \in D_v} \sigma$. Then

$$N_v \cdot \theta_{K/k,S} = 0$$

In particular, if there exists $v \in S$ totally split, then $\theta_{K/k,S} = 0$.

Let $\mathfrak{p}$ be a prime ideal of k not in S. Then

$$\theta_{K/k,S\cup\{\mathfrak{p}\}} = \theta_{K/k,S} \cdot (1 - \sigma_{\mathfrak{p}}^{-1})$$

For all $\xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, we have $\xi \cdot \theta_{K/k,S} \in \mathbb{Z}[G]$

Let $\mathscr{C}$ be the set of conjugacy classes of G.

Recall that $Z(\mathbb{C}[G]) = \mathbb{C}[C : C \in \mathscr{C}]$.

For $\chi \in \hat{G}$, the map $\Phi_\chi : Z(\mathbb{C}[G]) \to \mathbb{C}$ defined by $\Phi_\chi(C) = \dfrac{\chi(C)}{\chi(1)}$ is a (ring) homomorphism from $Z(\mathbb{C}[G])$ to $\mathbb{C}$.

The Brumer-Stickelberger element is the only element of $Z(\mathbb{C}[G])$ such that

$$\Phi_\chi(\theta_{K/k,S}) = L_{K/k,S}(0, \bar{\chi}), \ \forall \chi \in \hat{G}$$

Let $v \in S$ and let $N_v = \sum_{\sigma \in D_w} \frac{1}{C_\sigma} |C_\sigma| \in Z(\mathbb{C}[G])$ where $w \mid v$ and C_σ is the conjugacy class of σ. Then

$$N_v \cdot \theta_{K/k,S} = 0$$

In particular, if there exists $v \in S$ totally split, then $\theta_{K/k,S} = 0$.

Furthermore, for all complex conjugation τ in G, we also have

$$(1+\tau) \cdot \theta_{K/k,S} = 0$$

Let $\mathfrak{p}$ be a prime ideal of k not in S. Then

$$\theta_{K/k,S\cup\{\mathfrak{p}\}} = \theta_{K/k,S} \cdot \sum_{\chi \in \hat{G}} \det(1 - \rho_\chi(\sigma_\mathfrak{p}))\, e_{\bar{\chi}}$$

It follows from the principal rank zero Stark conjecture (proved by Tate) that

$$\theta_{K/k,S} \in \mathbb{Q}[G]$$

Explicit examples show that $w_K\, \theta_{K/k,S} \notin \mathbb{Z}[G]$ in general...

We make the following assumption:

Let $m_G := \mathrm{lcm}_{C\in\mathscr{C}}|C|$, then

$$m_G\, \xi \cdot \theta_{K/k,S} \in \mathbb{Z}[G],\ \forall \xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$$

Note that $m_G = 1$ if and only if G is abelian.

The Galois Brumer-Stark Conjecture $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$

We have

$$m_G w_K\, \theta_{K/k,S} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathsf{Cl}_K)$$

Furthermore, for any fractional ideal $\mathfrak{A}$ of K, there exists $\alpha \in K^\times$ with

- $\mathfrak{A}^{m_G w_K \theta_{K/k,S}} = (\alpha)$
- $\alpha \in K^\circ$
- ...

What about the "abelian condition"?

(Recall that it is conjectured that $K(\alpha^{1/w_K})/k$ is abelian when K/k is abelian.)

Consider G as a finite group. A group extension

$$1 \longrightarrow \Delta \longrightarrow \Gamma \xrightarrow{s} G \longrightarrow 1.$$

is central if $\Delta < Z(\Gamma)$.

Let

$$[\Gamma, \Gamma] := \langle \underbrace{\gamma_0 \gamma_1 \gamma_0^{-1} \gamma_1^{-1}}_{=:[\gamma_0, \gamma_1]} : \gamma_0, \gamma_1 \in \Gamma \rangle$$

be the commutator subgroup of Γ.

We say the above extension is strong central if $\Delta \cap [\Gamma, \Gamma] = 1$.

Let Γ be a group extension of G by Δ, that is

$$1 \longrightarrow \Delta \longrightarrow \Gamma \xrightarrow{s} G \longrightarrow 1.$$

We have

- If the extension is strong central then it is central.
  Proof. Let $\delta \in \Delta$ and $\gamma \in \Gamma$, $s([\gamma, \delta]) = 1$ thus $[\gamma, \delta] \in \Delta$ and $[\gamma, \delta] = 1$.
- The extension is strong central iff, for any $H < G$ with H abelian, $s^{-1}(H)$ is abelian.
  Proof. Assume strong central. For $\gamma, \gamma' \in s^{-1}(H)$, $s([\gamma, \gamma']) = 1$ so $[\gamma, \gamma'] = 1$ and $s^{-1}(H)$ is abelian. (Other direction: exercise!)
- If the extension is strong central then $m_\Gamma = m_G$.

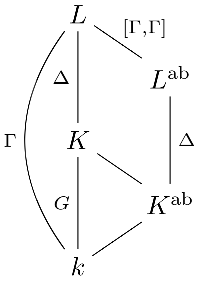
In particular, for Γ a strong central extension of G by Δ, the group Γ is abelian if and only if G is abelian.

Go back to our situation: K/k is a Galois extension with group G.

An extension L of K is a strong central extension of K/k if L/k is Galois and $\Gamma := \mathrm{Gal}(L/k)$ is a strong central extension of G by $\Delta := \mathrm{Gal}(L/K)$.

Let L^{ab} be the maximal sub-extension of L/k that is abelian over k. Then L is a strong central extension of K/k if and only if L/k is Galois and $L = KL^{\mathrm{ab}}$.

Furthermore, if L is strong central extension of K/k then $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(L^{\mathrm{ab}}/K^{\mathrm{ab}})$.



The Galois Brumer-Stark Conjecture $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$

We have

$$m_G w_K\,\theta_{K/k,S} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathsf{Cl}_K)$$

Furthermore, for any fractional ideal $\mathfrak{A}$ of K, there exists $\alpha \in K^\times$ with

- $\mathfrak{A}^{m_G w_K \theta_{K/k,S}} = (\alpha)$
- $\alpha \in K^\circ$
- $K(\alpha^{1/w_K})$ is a strong central extension of K/k

} $\mathbf{BS}_{\mathrm{Gal}}(K/k,S;\mathfrak{A})$

When K/k is abelian, we have $\mathbf{BS}(K/k,S) \iff \mathbf{BS}_{\mathrm{Gal}}(K/k,S)$.

For $\mathfrak{A}$ be a fractional ideal of K, $\mathbf{BS}_{\mathrm{Gal}}(K/k, S; \mathfrak{A})$ is equivalent to

1. There exists an extension L/K such that L is a strong central extension of K/k and $\gamma \in L^\circ$ with $(\mathfrak{A}\mathcal{O}_L)^{m_G\,\theta_{K/k,S}} = \gamma\mathcal{O}_L$.
2. For almost all prime ideals $\mathfrak{P}$ of K, there exists $\alpha_{\mathfrak{P}} \in K^\circ$ such that $\mathfrak{A}^{m_G(\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p}))\theta_{K/k,S}} = \alpha_{\mathfrak{P}}\mathcal{O}_K$ and $\alpha_{\mathfrak{P}} \equiv 1 \pmod{^* \mathfrak{Q}}$ for all $\mathfrak{Q} \mid \mathfrak{p}$ with $\sigma_{\mathfrak{Q}} = \sigma_{\mathfrak{P}}$.
3. For all $H < G$, abelian, there exist a family $(a_i)_{i\in I}$ of element of $\mathbb{Z}[H]$ generating $\mathrm{Ann}_{\mathbb{Z}[H]}(\mu_K)$ and a family $(\alpha_i)_{i\in I} \subset K^\circ$ such that $\mathfrak{A}^{m_G\,a_i\theta_{K/k,S}} = \alpha_i\mathcal{O}_K$ and ${\alpha_i}^{a_j} = {\alpha_j}^{a_i}$ for all $i, j \in I$.

The set $\{\mathfrak{A} : \mathbf{BS}_{\mathrm{Gal}}(K/k, S; \mathfrak{A}) \text{ is true}\}$ is a group stable under the action of G that contains the principal ideals of K.

The local Galois Brumer-Stark Conjecture $\mathbf{BS}^{(\ell)}_{\mathrm{Gal}}(K/k,S)$

Let $w_{K,\ell}$ be the ℓ-part of w_K. We have

$$m_G w_K\, \theta_{K/k,S} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathsf{Cl}_K\{\ell\})$$

Furthermore, for $\mathfrak{A}$ with $[\mathfrak{A}] \in \mathrm{Cl}_K\{\ell\}$, there exists $\alpha \in K^\times$ with

- $\mathfrak{A}^{m_G w_K \theta_{K/k,S}} = (\alpha)$
- $\alpha \in K^\circ$
- $K(\alpha^{1/w_{K,\ell}})$ is a strong central extension of K/k

We have $\mathbf{BS}_{\mathrm{Gal}}(K/k,S) \iff \mathbf{BS}^{(\ell)}_{\mathrm{Gal}}(K/k,S)\ \forall \ell$

Consider K'/k a Galois sub-extension of K/k with group G'.

Let ℓ such that one at least of the following conditions is true:

- $\ell \nmid w_K$,
- $m_G w_K \theta_{K'/k,S} \notin \ell\mathbb{Z}[G']$,
- there is no abelian sub-extension of $K/K'K^{\mathrm{ab}}$ of degree ℓ unramified outside of w_K.

Then

$$\mathbf{BS}^{(\ell)}_{\mathrm{Gal}}(K/k,S) \implies \tilde{\mathbf{BS}}^{(\ell)}_{\mathrm{Gal}}(K'/k,S)$$

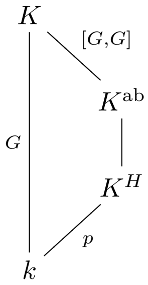
where $\tilde{\mathbf{BS}}^{(\ell)}_{\mathrm{Gal}}(K'/k,S)$ is $\mathbf{BS}^{(\ell)}_{\mathrm{Gal}}(K'/k,S)$ with $m_{G'}$ replaced by m_G.
(One proves easily that $m_{G'}$ divides m_G.)

We assume that there exists $H < G$, abelian, with $(G : H) = p$.

We have

$$\theta_{K/k,S} = \frac{1}{|[G,G]|}(\theta_{K^{\mathrm{ab}}/k,S} N_{K/K^{\mathrm{ab}}} - \theta_{K^{\mathrm{ab}}/K^H,S_H} N_{K/K^{\mathrm{ab}}}) + \theta_{K/K^H,S_H}$$

where $S_H := \{w \in \mathrm{Pl}(K^H) : w_{|k} \in S\}$.



In this case, $|[G,G]| \mid m_G$, thus $m_G\,\xi \cdot \theta_{K/k,S} \in \mathbb{Z}[G]$, for all $\xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$.

Recall that

$$\theta_{K/k,S} = \frac{1}{|[G,G]|}(\theta_{K^{\mathrm{ab}}/k,S} N_{K/K^{\mathrm{ab}}} - \theta_{K^{\mathrm{ab}}/K^H,S_H} N_{K/K^{\mathrm{ab}}}) + \theta_{K/K^H,S_H}$$

If H is of odd order then K^H is not totally real and

$$\theta_{K^{\mathrm{ab}}/K^H,S_H} = \theta_{K/K^H,S_H} = 0.$$

We prove that $\mathbf{BS}(K^{\mathrm{ab}}/k,S) \implies \mathbf{BS}_{\mathrm{Gal}}(K/k,S)$.

Consequence. $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ is true if $G \simeq D_n$ with n odd.

Recall that

$$\theta_{K/k,S} = \frac{1}{|[G,G]|}(\theta_{K^{\mathrm{ab}}/k,S} N_{K/K^{\mathrm{ab}}} - \theta_{K^{\mathrm{ab}}/K^H,S_H} N_{K/K^{\mathrm{ab}}}) + \theta_{K/K^H,S_H}$$

Assume H is of even order and that $\mathbf{BS}(K^{\mathrm{ab}}/k,S)$ and $\mathbf{BS}(K/K^H,S_H)$ hold (thus also $\mathbf{BS}(K^{\mathrm{ab}}/K^H,S)$). Then

$$m_G w_K\, \theta_{K/k,S} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathsf{Cl}_K)$$

And, for any fractional ideal $\mathfrak{A}$ of K, there exists $\alpha \in K^\times$ with

- $\mathfrak{A}^{m_G w_K \theta_{K/k,S}} = (\alpha)$
- $\alpha \in K^\circ$
- $K(\alpha^{1/w_K})/K^H$ is abelian

Consequence. $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ is true if G is non-abelian of order 8.